Applications of the hyperbolic Ax-Schanuel conjecture

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Christopher Daw Jinbo Ren

Abstract

In 2014, Pila and Tsimerman gave a proof of the Ax-Schanuel conjecture for the j-function and, with Mok, have recently announced a proof of its generalization to any (pure) Shimura variety. We refer to this generalization as the hyperbolic Ax-Schanuel conjecture. In this article, we show that the hyperbolic Ax-Schanuel conjecture can be used to reduce the Zilber-Pink conjecture for Shimura varieties to a problem of point counting. We further show that this point counting problem can be tackled in a number of cases using the Pila-Wilkie counting theorem and several arithmetic conjectures. Our methods are inspired by previous applications of the Pila-Zannier method and, in particular, the recent proof by Habegger and Pila of the Zilber-Pink conjecture for curves in abelian varieties.

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Contents

1 Introduction

The Ax-Schanuel theorem [2] is a result regarding the transcendence degrees of fields generated over the complex numbers by power series and their exponentials. Formulated geometrically for the uniformization maps of algebraic tori, it has inspired analogous statements for the uniformization maps of abelian varieties and Shimura varieties. The former, following from another theorem of Ax [3], has recently been used by Habegger and Pila in their proof of the Zilber-Pink conjecture for curves in abelian varieties [18].

Habegger and Pila also extended the Pila-Zannier strategy to the Zilber-Pink conjecture for products of modular curves. Their method relies on an Ax-Schanuel conjecture for the j-function and is conditional on their so-called large Galois orbits conjecture. The purpose of this paper is to show that the Pila-Zannier strategy can be extended to the Zilber-Pink conjecture for general Shimura varieties.

This conjecture can just as easily be stated in the generality of **mixed** Shimura varieties but, in this article, we will restrict our attention to **pure** Shimura varieties, though we have no explicit reason to believe that the methods presented here will not extend to the mixed setting. We begin by stating a conjecture of Pink.

Conjecture 1.1 (cf. [28], Conjecture 1.3)**.** *Let $\mathrm{Sh}_K(G,\mathfrak{X})$ be a Shimura variety and, for any integer d, let $\mathrm{Sh}_K(G,\mathfrak{X})^{[d]}$ denote the union of the special subvarieties of $\mathrm{Sh}_K(G,\mathfrak{X})$ having codimension at least d. Let V be a* **Hodge generic** *subvariety of $\mathrm{Sh}_K(G,\mathfrak{X})$. Then*

$$V \cap \mathrm{Sh}_K(G,\mathfrak{X})^{[1+\dim V]}$$

is not Zariski dense in V.

The heuristics of this conjecture are as follows. For two subvarieties V and W of $\mathrm{Sh}_K(G,\mathfrak{X})$, such that the codimension of W is at least $1+\dim V$, we expect $V\cap W=\emptyset$. Even if we fix V and take the union of $V\cap W$ for countably many W of codimension at least $1+\dim V$, the resulting set should still be rather small in V unless, of course, V was not sufficiently generic in $\mathrm{Sh}_K(G,\mathfrak{X})$. Pink's conjecture turns this expectation into an explicit statement about the intersection of Hodge generic subvarieties with the special subvarieties of small dimension.

Conjecture 1.1 can also be formulated for algebraic tori, abelian varieties, or even semi-abelian varieties, though Conjecture 1.1 for mixed Shimura varieties implies all of these formulations (see [28]). When V is a curve, defined over a number field, and contained in an algebraic torus, we obtain a theorem of Maurin [21]. When V is a curve, defined over a number field, and contained in an abelian variety, we obtain the recent theorem of Habegger and Pila [18], and it is the ideas presented there that form the basis for this article. Habegger and Pila had given some partial results when V is a curve, defined over a number field, and contained in the Shimura variety $\mathbb{C}^n$ [17], and Orr has recently generalized their results to a curve contained in $\mathcal{A}_g^2$ (see [25] for more details).

We should point out that Conjecture 1.1 implies the André-Oort conjecture.

Conjecture 1.2 (André-Oort)**.** *Let $\mathrm{Sh}_K(G,\mathfrak{X})$ be a Shimura variety and let V be a subvariety of $\mathrm{Sh}_K(G,\mathfrak{X})$ such that the special points of $\mathrm{Sh}_K(G,\mathfrak{X})$ in V are Zariski dense in V. Then V is a special subvariety of $\mathrm{Sh}_K(G,\mathfrak{X})$.*

To see this, we may assume that V is Hodge generic in $\mathrm{Sh}_K(G,\mathfrak{X})$. Then, since special points have codimension $\dim \mathrm{Sh}_K(G,\mathfrak{X})$, Conjecture 1.1 implies that, either $\dim V=\dim \mathrm{Sh}_K(G,\mathfrak{X})$, in which case V is a connected component of $\mathrm{Sh}_K(G,\mathfrak{X})$ and, in particular, a special subvariety of $\mathrm{Sh}_K(G,\mathfrak{X})$, or the set of special points of $\mathrm{Sh}_K(G,\mathfrak{X})$ in V are not Zariski dense in V.

In precisely the same fashion, the Zilber-Pink conjecture for abelian varieties implies the Manin-Mumford conjecture.

The André-Oort conjecture has a rich history of its own. Here, we simply recall that it was recently settled for $\mathcal{A}_g$ by Pila and Tsimerman [26, 30], thanks to recent progress on the Colmez conjecture due to Andreatta, Goren, Howard, Madapusi Pera, Yuan, and Zhang [1, 40], and it is known to hold for all Shimura varieties under conjectural lower bounds for Galois orbits of special points due to the work of Orr, Klingler, Ulmo, Yafaev, and the first author [8, 20, 37]. Furthermore, Gao has generalized these proofs to all mixed Shimura varieties [13, 14].

In his work on Schanuel's conjecture, Zilber made his own conjecture on unlikely intersections [41], which was closely related to the independent work of Bombieri, Masser, and Zannier [5]. To describe their formulations, we require the following definition.

Definition 1.3. Let $\mathrm{Sh}_K(G,\mathfrak{X})$ be a Shimura variety and let V be a subvariety of $\mathrm{Sh}_K(G,\mathfrak{X})$. A subvariety W of V is called **atypical** with respect to V if there is a special subvariety T of $\mathrm{Sh}_K(G,\mathfrak{X})$ such that W is an irreducible component of $V\cap T$ and

$$\dim W > \dim V + \dim T - \dim \mathrm{Sh}_K(G,\mathfrak{X}).$$

We denote by $\mathrm{Atyp}(V)$ the union of the subvarieties of V that are atypical with respect to V.

Zilber's conjecture, formulated for Shimura varieties, is then as follows.

Conjecture 1.4 (cf. [18], Conjecture 2.2)**.** *Let $\mathrm{Sh}_K(G,\mathfrak{X})$ be a Shimura variety and let V be a subvariety of $\mathrm{Sh}_K(G,\mathfrak{X})$. Then $\mathrm{Atyp}(V)$ is equal to a finite union of subvarieties of V.*

We will see that Conjecture 1.4 strengthens Conjecture 1.1 and, therefore, it is Conjecture 1.4 that we refer to as the **Zilber-Pink** conjecture. Habegger and Pila obtained a proof of the Zilber-Pink conjecture for products of modular curves assuming the weak complex Ax conjecture and the large Galois orbits conjecture. Subsequently, Pila and Tsimerman obtained the weak complex Ax conjecture as a corollary to their proof of the Ax-Schanuel conjecture for the j-function [27]. Habegger and Pila had previously verified the large Galois orbits conjecture for so-called asymmetric curves [17].

This article seeks to generalize the ideas of [18] to general Shimura varieties. Hence, we will have to make generalizations of the previously mentioned hypotheses. The foremost of which will be the statement from functional transcendence, namely, the hyperbolic Ax-Schanuel conjecture that generalizes the Ax-Schanuel conjecture for the j-function to general Shimura varieties. It is with this ingredient that we prove our main result (Theorem 8.3), that, under the hyperbolic Ax-Schanuel conjecture, the Zilber-Pink conjecture can be reduced to a problem of point counting. However, given that Mok, Pila, and Tsimerman have recently announced a proof of the hyperbolic Ax-Schanuel conjecture, this result is now unconditional. Besides than the hyperbolic Ax-Schanuel conjecture, our main input will be the theory of o-minimality and, in particular, the fact that the uniformization map of a Shimura variety is definable in $\mathbb{R}_{\mathrm{an,exp}}$ when restricted to an appropriate fundamental domain.

After establishing the main result, we attempt to tackle the point counting problem using the now famous Pila-Wilkie counting theorem. To do so, we formulate several arithmetic conjectures that are inspired by previous applications of the Pila-Zannier strategy. In this vein, our paper is very much in the spirit of [33], which, at the time, reduced the André-Oort conjecture to a point counting problem and then explained how various conjectural ingredients, namely, the hyperbolic Ax-Lindemann conjecture, lower bounds for Galois orbits of special points, upper bounds for the heights of pre-special points, and the definability of the uniformization map, could be combined to deliver a proof of the André-Oort conjecture.

Our arithmetic hypotheses are (1) lower bounds for Galois orbits of so-called optimal points, which we also refer to as the large Galois orbits conjecture, and (2) upper bounds for the heights of pre-special subvarieties. Hypothesis (1) generalizes the (in some cases still conjectural) lower bounds for Galois orbits of special points (when such special points are also maximal special subvarieties), and also generalizes the large Galois orbits conjecture of Habegger and Pila. Hypothesis (2) generalizes the upper bounds for heights of pre-special points, which were proved by Orr and the first author [8]. However, we also show that it is possible to replace hypothesis (2) with two other arithmetic hypotheses, namely, (3) upper bounds for the degrees of fields associated with special subvarieties, and (4) upper bounds for the heights of lattice elements. Hypothesis (3) is a replacement for the fact that, for an abelian variety, its abelian subvarieties can be defined over a fixed finite extension of the base field. Hypothesis (4) is an analogue of a known result for abelian varieties. We verify hypotheses (2), (3), and (4) for a product of modular curves.

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Conventions

- ⋆ Throughout this paper, **definable** means definable in the o-minimal structure $\mathbb{R}_{\mathrm{an,exp}}$.
- ⋆ Unless preceeded by the word **Shimura**, a variety we will mean a **geometrically irreducible** variety.

Index of notations We collect here the main symbols appearing in this article.

- ⋆ $\langle W\rangle$ is the smallest special subvariety containing W.
- ⋆ $\langle W\rangle_{\mathrm{ws}}$ is the smallest weakly-special subvariety containing W.
- ⋆ $\langle A\rangle_{\mathrm{Zar}}$ is the smallest algebraic subvariety containing A.
- ⋆ $\langle A\rangle_{\mathrm{geo}}$ is the smallest totally geodesic subvariety containing A.
- ⋆ $\delta(W) := \dim\langle W\rangle - \dim W$
- ⋆ $\delta_{\mathrm{ws}}(W) := \dim\langle W\rangle_{\mathrm{ws}} - \dim W$
- ⋆ $\delta_{\mathrm{Zar}}(A) := \dim\langle A\rangle_{\mathrm{Zar}} - \dim A$
- ⋆ $\delta_{\mathrm{geo}}(A) := \dim\langle A\rangle_{\mathrm{geo}} - \dim A$

* $\mathrm{Opt}(V)$ is the set of subvarieties of V that are optimal in V.
* $\mathrm{Opt}_0(V)$ is the set of points of V that are optimal in V.
* G^{ad} is the adjoint group of G i.e. the quotient of G by its centre.
* G^{der} is derived group of G.
* G° is the connected component of G containing the identity.
* $G_H := H \cdot Z_G(H)^{\circ}$ whenever H is a subgroup of G.

2 Special and weakly special subvarieties

Let $(G, \mathfrak{X})$ be a Shimura datum and let K be a compact open subgroup of $G(\mathbb{A}_f)$, where $\mathbb{A}_f$ will henceforth denote the finite rational adèles. Let $\mathrm{Sh}_K(G, \mathfrak{X})$ denote the corresponding **Shimura variety**. By this, we mean the complex quasi-projective algebraic variety such that $\mathrm{Sh}_K(G, \mathfrak{X})(\mathbb{C})$ is equal to the image of

$$G(\mathbb{Q})\backslash[\mathfrak{X} \times (G(\mathbb{A}_f)/K)] \tag{2.0.1}$$

under the canonical embedding into complex projective space given by Baily and Borel [4]. We will identify (2.0.1) with $\mathrm{Sh}_K(G, \mathfrak{X})(\mathbb{C})$. We recall that, on $\mathfrak{X} \times (G(\mathbb{A}_f)/K)$, the action of $G(\mathbb{Q})$ is the diagonal one.

Let X be a connected component of $\mathfrak{X}$ and let $G(\mathbb{Q})_+$ be the subgroup of $G(\mathbb{Q})$ acting on it. For any $g \in G(\mathbb{A}_f)$, we obtain a **congruence** subgroup Γ_g of $G(\mathbb{Q})_+$ by intersecting it with gKg^{-1}. Furthermore, the locally symmetric variety $\Gamma_g\backslash X$ is contained in (2.0.1) via the map that sends the class of x to the class of (x, g). If we take the disjoint union of the $\Gamma_g\backslash X$ over a (finite) set of representatives for

$$G(\mathbb{Q})_+\backslash G(\mathbb{A}_f)/K,$$

the corresponding inclusion map is a bijection.

Definition 2.1. For any compact open subgroup K' of $G(\mathbb{A}_f)$ contained in K, we obtain a finite morphism

$$\mathrm{Sh}_{K'}(G, \mathfrak{X}) \to \mathrm{Sh}_K(G, \mathfrak{X}),$$

given by the natural projection. Furthermore, for any $a \in G(\mathbb{A}_f)$, we obtain an isomorphism

$$\mathrm{Sh}_K(G, \mathfrak{X}) \to \mathrm{Sh}_{a^{-1}Ka}(G, \mathfrak{X})$$

sending the class of (x, g) to the class of (x, ga). We let $T_{K,a}$ denote the map on algebraic cycles of $\mathrm{Sh}_K(G, \mathfrak{X})$ given by the algebraic correspondence

$$\mathrm{Sh}_K(G, \mathfrak{X}) \leftarrow \mathrm{Sh}_{K\cap aKa^{-1}}(G, \mathfrak{X}) \to \mathrm{Sh}_{a^{-1}Ka\cap K}(G, \mathfrak{X}) \to \mathrm{Sh}_K(G, \mathfrak{X}),$$

where the outer arrows are the natural projections and the middle arrow is the isomorphism given by a. We refer to a map of this sort as a **Hecke correspondence**.

Definition 2.2. Let $(H, \mathfrak{X}_H)$ be a Shimura subdatum of $(G, \mathfrak{X})$ and let K_H denote a compact open subgroup of $H(\mathbb{A}_f)$ contained in K. The natural map

$$H(\mathbb{Q})\backslash[\mathfrak{X}_H \times (H(\mathbb{A}_f)/K_H)] \to G(\mathbb{Q})\backslash[\mathfrak{X} \times (G(\mathbb{A}_f)/K)]$$

yields a closed morphism of Shimura varieties

$$\mathrm{Sh}_{K_H}(H, \mathfrak{X}_H) \to \mathrm{Sh}_K(G, \mathfrak{X})$$

and we refer to the image of any such morphism as a **Shimura subvariety** of $\mathrm{Sh}_K(G, \mathfrak{X})$.

For any Shimura subvariety Z of $\mathrm{Sh}_K(G, \mathfrak{X})$ and any $a \in G(\mathbb{A}_f)$, we refer to any irreducible component of $T_{K,a}(Z)$ as a **special subvariety** of $\mathrm{Sh}_K(G, \mathfrak{X})$.

Recall that, by definition, $\mathfrak{X}$ is a $G(\mathbb{R})$ conjugacy class of morphisms from $\mathbb{S}$ to $G_{\mathbb{R}}$ and the **Mumford-Tate group** $\mathrm{MT}(x)$ of $x \in \mathfrak{X}$ is defined as the smallest $\mathbb{Q}$-subgroup H of G such that x factors through $H_{\mathbb{R}}$. If we let $\mathfrak{X}_M$ denote the $M(\mathbb{R})$ conjugacy class of $x \in X$, where $M := \mathrm{MT}(x)$, then $(M, \mathfrak{X}_M)$ is a Shimura subdatum of $(G, \mathfrak{X})$. In particular, if we let X_M denote a connected component of $\mathfrak{X}_M$ contained in X, then the image of X_M in $\Gamma_g\backslash X$, for any $g \in G(\mathbb{A}_f)$, is a special subvariety of $\mathrm{Sh}_K(G, \mathfrak{X})$, and it is easy to see that every special subvariety of $\mathrm{Sh}_K(G, \mathfrak{X})$ arises this way.

Of course, if $x \in X_M$, then X_M is equal to the $M(\mathbb{R})^+$ conjugacy class of x. Furthermore, the action of $M(\mathbb{R})$ on $\mathfrak{X}_M$ factors through $M^{\mathrm{ad}}(\mathbb{R})$ and the group M^{ad} is equal to the direct product of its $\mathbb{Q}$-simple factors. Therefore, we can write M^{ad} as a product

$$M^{\mathrm{ad}} = M_1 \times M_2$$

of two normal $\mathbb{Q}$-subgroups, either of which may (by choice or necessity) be trivial, and we thus obtain a corresponding splitting

$$X_M = X_1 \times X_2.$$

For any such splitting, and any $x_1 \in X_1$ or $x_2 \in X_2$, we refer to the image of $\{x_1\} \times X_2$ or $X_1 \times \{x_2\}$ in $\Gamma_g\backslash X$, for any $g \in G(\mathbb{A}_f)$, as a **weakly special subvariety** of $\mathrm{Sh}_K(G, \mathfrak{X})$. In particular, every special subvariety of $\mathrm{Sh}_K(G, \mathfrak{X})$ is a weakly special subvariety of $\mathrm{Sh}_K(G, \mathfrak{X})$. By [23], Section 4, the weakly special subvarieties of $\mathrm{Sh}_K(G, \mathfrak{X})$ are precisely those subvarieties of $\mathrm{Sh}_K(G, \mathfrak{X})$ that are totally geodesic in $\mathrm{Sh}_K(G, \mathfrak{X})$. Furthermore, a weakly special subvariety of $\mathrm{Sh}_K(G, \mathfrak{X})$ is a special subvariety of $\mathrm{Sh}_K(G, \mathfrak{X})$ if and only if it contains a special subvariety of dimension zero, henceforth known as a **special point**.

Remark 2.3. The following observations will facilitate various reductions.

- ⋆ Let K' be a compact open subgroup of $G(\mathbb{A}_f)$ contained in K. By definition, a subvariety Z of $\mathrm{Sh}_K(G, \mathfrak{X})$ is a (weakly) special subvariety of $\mathrm{Sh}_K(G, \mathfrak{X})$ if and only if any irreducible component of the inverse image of Z in $\mathrm{Sh}_{K'}(G, \mathfrak{X})$ is a (weakly) special subvariety of $\mathrm{Sh}_{K'}(G, \mathfrak{X})$.
- ⋆ For any $a \in G(\mathbb{A}_f)$, a subvariety Z of $\mathrm{Sh}_K(G, \mathfrak{X})$ is a (weakly) special subvariety of $\mathrm{Sh}_K(G, \mathfrak{X})$ if and only if any irreducible component of $T_{K,a}(Z)$ is a (weakly) special subvariety of $\mathrm{Sh}_{K'}(G, \mathfrak{X})$.
- ⋆ If we let G^{ad} denote the adjoint group of G i.e. the quotient of G by its centre, we obtain another Shimura datum $(G^{\mathrm{ad}}, \mathfrak{X}^{\mathrm{ad}})$, known as the **adjoint Shimura datum** associated

with $(G, \mathfrak{X})$. For any compact open subgroup K^{ad} of $G^{\mathrm{ad}}(\mathbb{A}_f)$ containing the image of K, we obtain a finite morphism

$$\mathrm{Sh}_K(G, \mathfrak{X}) \to \mathrm{Sh}_{K^{\mathrm{ad}}}(G^{\mathrm{ad}}, \mathfrak{X}^{\mathrm{ad}}).$$

As in [11], Proposition 2.2, a subvariety Z of $\mathrm{Sh}_K(G, \mathfrak{X})$ is a (weakly) special subvariety of $\mathrm{Sh}_K(G, \mathfrak{X})$ if and only if any irreducible component of the inverse image of Z in $\mathrm{Sh}_{K^{\mathrm{ad}}}(G^{\mathrm{ad}}, \mathfrak{X}^{\mathrm{ad}})$ is a (weakly) special subvariety of $\mathrm{Sh}_{K^{\mathrm{ad}}}(G^{\mathrm{ad}}, \mathfrak{X}^{\mathrm{ad}})$.

Lemma 2.4. *Let S, T be two totally geodesic submanifolds of a Riemannian manifold M. If $S \cap T$ is non-empty then it is totally geodesic.*

Proof. Assume $S \cap T$ is non-empty and let p be a point on $S \cap T$. Let

$$v \in T_p(S \cap T) \subseteq T_p(S) \cap T_p(T)$$

and let $\gamma : (-1, 1) \to M$ denote the geodesic on M such that $\gamma'(0) = v$. Since S and T are totally geodesic, there exist $0 < \epsilon_S, \epsilon_T < 1$ such that $\gamma(-\epsilon_S, \epsilon_S) \subseteq S$ and $\gamma(-\epsilon_T, \epsilon_T) \subseteq T$. Hence, if $\epsilon := \min\{\epsilon_S, \epsilon_T\} > 0$, then

$$\gamma(-\epsilon, \epsilon) \subseteq S \cap T,$$

proving the claim. □

Corollary 2.5. *The intersection of two weakly special subvarieties of* $\mathrm{Sh}_K(G, \mathfrak{X})$ *is either empty or equal to a finite union of weakly special subvarieties of* $\mathrm{Sh}_K(G, \mathfrak{X})$.

Proof. Without loss of generality, we may assume that K is neat and, hence, torsion free. In particular, $\mathrm{Sh}_K(G, \mathfrak{X})$ is naturally endowed with the structure of a Riemannian manifold. The intersection of two subvarieties of $\mathrm{Sh}_K(G, \mathfrak{X})$ is either empty or equal to a finite union of subvarieties of $\mathrm{Sh}_K(G, \mathfrak{X})$, which, by Lemma 2.4, are totally geodesic in $\mathrm{Sh}_K(G, \mathfrak{X})$. □

Therefore, for any subvariety W of $\mathrm{Sh}_K(G, \mathfrak{X})$, there exists a **smallest** weakly special subvariety $\langle W \rangle_{\mathrm{ws}}$ of $\mathrm{Sh}_K(G, \mathfrak{X})$ containing W. We note that here, and throughout, our notations and terminology regarding subvarieties often differ from those found in [18].

Corollary 2.6. *The intersection of two special subvarieties Z_1 and Z_2 of* $\mathrm{Sh}_K(G, \mathfrak{X})$ *is either empty or equal to a finite union of special subvarieties of* $\mathrm{Sh}_K(G, \mathfrak{X})$.

Proof. By Corollary 2.5, the intersection of two special subvarieties of $\mathrm{Sh}_K(G, \mathfrak{X})$ is either empty or equal to a finite union of weakly special subvarieties of $\mathrm{Sh}_K(G, \mathfrak{X})$.

Assume the intersection is not empty. There exist Shimura subdata $(H_1, \mathfrak{X}_1)$ and $(H_2, \mathfrak{X}_2)$, connected components X_1 and X_2 of $\mathfrak{X}_1$ and $\mathfrak{X}_2$, respectively, contained in X, and a $g \in G(\mathbb{A}_f)$ such that Z_1 and Z_2 are equal to the images of X_1 and X_2 in $\Gamma_g \backslash X$, respectively. Let Z denote an irreducible component of the intersection and let $x \in X$ be a point whose image in $\Gamma_g \backslash X$ belongs to Z. There exist $\gamma_1, \gamma_2 \in \Gamma_g$ such that $M := \mathrm{MT}(x)$ is contained in

$$\gamma_1 H_1 \gamma_1^{-1} \cap \gamma_2 H_2 \gamma_2^{-1}.$$

Hence, the image of the $M(\mathbb{R})^+$ conjugacy class of x is contained in Z and this is a special subvariety of $\mathrm{Sh}_K(G, \mathfrak{X})$. Therefore, Z contains a special point of $\mathrm{Sh}_K(G, \mathfrak{X})$ and we conclude that Z is a special subvariety of $\mathrm{Sh}_K(G, \mathfrak{X})$. □

Therefore, for any subvariety W of $\mathrm{Sh}_K(G, \mathfrak{X})$, there exists a **smallest** special subvariety $\langle W \rangle$ of $\mathrm{Sh}_K(G, \mathfrak{X})$ containing W.

3 The Zilber-Pink conjecture

For the remainder of this article, we fix a Shimura datum $(G, \mathfrak{X})$ and we let X be a connected component of $\mathfrak{X}$. We fix a compact open subgroup K of $G(\mathbb{A}_f)$ and we let

$$\Gamma := G(\mathbb{Q})_+ \cap K,$$

where $G(\mathbb{Q})_+$ is the subgroup of $G(\mathbb{Q})$ acting on X. We denote by S the variety $\Gamma \backslash X$.

As in [18], we will consider an equivalent formulation of Conjecture 1.4 using the language of optimal subvarieties.

Definition 3.1. Let W be a subvariety of S. We define the **defect** of W to be

$$\delta(W) := \dim \langle W \rangle - \dim W.$$

Definition 3.2. Let V be a subvariety of S and let W be a subvariety of V. Then W is called **optimal** in V if, for any subvariety Y of S,

$$W \subsetneq Y \subseteq V \implies \delta(Y) > \delta(W).$$

We denote by $\mathrm{Opt}(V)$ the set of all subvarieties of V that are optimal in V.

Remark 3.3. Let V be a subvariety of S. First note that $V \in \mathrm{Opt}(V)$. Secondly, if $W \in \mathrm{Opt}(V)$, then W is a component of

$$\langle W \rangle \cap V.$$

Conjecture 3.4 (cf. [18], Conjecture 2.6)**.** *Let V be a subvariety of S. Then* $\mathrm{Opt}(V)$ *is finite.*

Observe that a maximal special subvariety of V is an optimal subvariety of V. Therefore, Conjecture 3.4 immediately implies that V contains only finitely many maximal special subvarieties, which is another formulation of the André-Oort conjecture for V.

Lemma 3.5. *The Zilber-Pink conjecture (Conjecture 1.4) is equivalent to Conjecture 3.4.*

Proof. Consider the situation described in the statement of Conjecture 1.4. By Remark 8, we suffer no loss in generality if we assume that V is contained in S. Then the result follows from [18], Lemma 2.7. ☐

Lemma 3.6. *The Zilber-Pink conjecture implies Conjecture 1.1.*

Proof. By Lemma 3.5, it suffices to show that Conjecture 3.4 implies Conjecture 1.1.

Consider the situation described in Conjecture 1.1. By Remark 8, we suffer no loss in generality if we assume that V is contained in S. Let P be a point belonging to

$$V \cap \mathrm{Sh}_K(G, \mathfrak{X})^{[1+\dim V]}.$$

Let W be a subvariety of V that is optimal in V and contains P such that

$$\delta(W) \leq \delta(P) = \dim\langle P \rangle.$$

Since P belongs to a special subvariety of codimension at least $\dim V + 1$ and V is Hodge generic in $\mathrm{Sh}_K(G, \mathfrak{X})$, we have

$$\dim\langle P \rangle \leq \dim S - \dim V - 1 = \dim\langle V \rangle - \dim V - 1 < \delta(V).$$

Therefore, $\delta(W) < \delta(V)$ and we conclude that W is not V.

According to Conjecture 3.4, the union of the subvarieties belonging to $\mathrm{Opt}(V) \setminus V$ is not Zariski dense in V. ☐

4 The defect condition

In this section, we prove Habegger and Pila's defect condition for Shimura varieties and thus show that a subvariety that is optimal is weakly optimal.

Definition 4.1. Let W be a subvariety of S. We define the **weakly special defect** of W to be

$$\delta_{\mathrm{ws}}(W) := \dim\langle W\rangle_{\mathrm{ws}} - \dim W.$$

We note that, in [18], this notion was referred to as geodesic defect.

Definition 4.2. If V is a subvariety of S and W a subvariety of V, then W is called **weakly optimal** in V if, for any subvariety Y of S,

$$W \subsetneq Y \subseteq V \implies \delta_{\mathrm{ws}}(Y) > \delta_{\mathrm{ws}}(W).$$

Remark 4.3. Let V be a subvariety of S and W a subvariety of V. If W is weakly optimal in V, then W is a component of

$$\langle W\rangle_{\mathrm{ws}} \cap V.$$

Proposition 4.4 (cf. [18], Proposition 4.3)**.** *The following* **defect condition** *holds.*

Let $W \subseteq Y$ be two subvarieties of S. Then

$$\delta(Y) - \delta_{\mathrm{ws}}(Y) \leq \delta(W) - \delta_{\mathrm{ws}}(W).$$

Proof. We need to show that

$$\dim\langle Y\rangle - \dim\langle Y\rangle_{\mathrm{ws}} \leq \dim\langle W\rangle - \dim\langle W\rangle_{\mathrm{ws}}.$$

By Remark 8, we can and do assume that G is the generic Mumford-Tate group on X, that it is equal to G^{ad}, and that Y is Hodge generic in S. By definition, there exists a decomposition

$$G = G_1 \times G_2,$$

which induces a splitting

$$X = X_1 \times X_2,$$

such that $\langle Y\rangle_{\mathrm{ws}}$ is equal to the image of $X_1 \times \{x_2\}$ in S, for some $x_2 \in X_2$.

Let $\Gamma_1 := \mathrm{p}_1(\Gamma)$ and $\Gamma_2 := \mathrm{p}_2(\Gamma)$, where p_1 and p_2 are the projections from G to G_1 and G_2, respectively. Then $\Gamma' := \Gamma_1 \times \Gamma_2$ is a congruence subgroup of $G(\mathbb{Q})_+$ containing Γ as a finite index subgroup. Let $\phi : \Gamma\backslash X \to \Gamma'\backslash X$ denote the natural (finite) morphism. Then $\phi(W) \subseteq \phi(Y) \subseteq S' := \Gamma'\backslash X$, and we have

$$\begin{aligned}
\dim\langle Y\rangle &= \dim\langle\phi(Y)\rangle,\\
\dim\langle W\rangle &= \dim\langle\phi(W)\rangle,\\
\dim\langle Y\rangle_{\mathrm{ws}} &= \dim\langle\phi(Y)\rangle_{\mathrm{ws}},\\
\dim\langle W\rangle_{\mathrm{ws}} &= \dim\langle\phi(W)\rangle_{\mathrm{ws}}.
\end{aligned}$$

Therefore, after replacing Y, W, and S by $\phi(Y)$, $\phi(W)$, and S', respectively, we may assume that Γ is of the form $\Gamma_1 \times \Gamma_2$, and $S = \Gamma_1\backslash X_1 \times \Gamma_2\backslash X_2 = S_1 \times S_2$.

Thus, $\langle Y\rangle_{\rm ws}=S_1\times\{s_2\}$, where s_2 is the image of x_2 in S_2, $Y=Y_1\times\{s_2\}$, where Y_1 is the projection of Y to S_1, and $W=W_1\times\{s_2\}$, where W_1 is the projection of W to S_1. In particular, we can take

$$x:=(x_1,x_2)\in X_1\times X_2$$

such that $\langle W\rangle$ is equal to the image in S of the $M(\mathbb{R})^+$ conjugacy class X_M of x, where $M:=\mathrm{MT}(x)$.

Again, there exists a decomposition

$$M^{\rm ad}=M_1\times M_2,$$

which induces a splitting

$$X_M=X_{M_1}\times X_{M_2}\subseteq X=X_1\times X_2$$

such that $\langle W\rangle_{\rm ws}$ is equal to the image in S of $X_{M_1}\times\{y_2\}$, for some $y_2\in X_{M_2}$.

Since $\mathrm{MT}(x_2)$ is equal to G_2, it follows that M is a subgroup of $G_1\times G_2$ that surjects on to the second factor. In particular,

$$X_M=M^{\rm der}(\mathbb{R})^+x$$

surjects on to X_2. Therefore, let M_1' and M_2' be two normal semisimple subgroups of $M^{\rm der}$ corresponding to M_1 and M_2, respectively, so that

$$M^{\rm der}(\mathbb{R})^+x=M_1'(\mathbb{R})^+M_2'(\mathbb{R})^+x.$$

Since W is contained in $S_1\times\{s_2\}$, the projection of M_1' to G_2 must be trivial. Hence, $M_1'(\mathbb{R})^+x$ is contained in $X_1\times\{x_2\}$ and we conclude that $M_2'(\mathbb{R})^+x$ surjects on to X_2. Since

$$M_2'(\mathbb{R})^+x=\{y_1\}\times X_{M_2},$$

for some $y_1\in X_{M_1}$, we have

$$\dim\langle W\rangle-\dim\langle W\rangle_{\rm ws}=\dim X_{M_2}\geq\dim X_2=\dim\langle Y\rangle-\dim\langle Y\rangle_{\rm ws},$$

as required. □

Corollary 4.5 (cf. [18], Proposition 4.5). *Let V be a subvariety of S. A subvariety of V that is optimal in V is weakly optimal in V.*

5 The hyperbolic Ax-Schanuel conjecture

In this section, we formulate various conjectures about Shimura varieties that are analogous to the original Ax-Schanuel theorem from functional transcendence theory.

Theorem 5.1 (cf. [2], Theorem 1). *Let $f_1,...,f_n\in\mathbb{C}[[t_1,...,t_m]]$ be power series that are $\mathbb{Q}$-linearly independent modulo $\mathbb{C}$. Then we have the following inequality*

$$\mathrm{tr.deg}_{\mathbb{C}}\mathbb{C}(f_1,...,f_n,e(f_1),...,e(f_n))\geq n+\mathrm{rank}(\frac{\partial f_i}{\partial t_j})_{\substack{i=1,...,n\\ j=1,...,m}}$$

where $e(f)=e^{2\pi i f}$.

The following theorem is then an immediate corollary.

Theorem 5.2. *Let $f_1, ..., f_n \in \mathbb{C}[[t_1, ..., t_m]]$ as above. Then*

$$\text{tr.deg}_{\mathbb{C}}\mathbb{C}(f_1, ..., f_n) + \text{tr.deg}_{\mathbb{C}}\mathbb{C}(e(f_1), ..., e(f_n)) \geq n + \text{rank}(\frac{\partial f_i}{\partial t_j})_{\substack{i=1,...,n \\ j=1,...,m}}.$$

Let π denote the uniformization map

$$\mathbb{C}^n \to (\mathbb{C}^\times)^n : (x_1, ..., x_n) \mapsto (e(x_1), ..., e(x_n))$$

and let D_n denote its graph in $\mathbb{C}^n \times (\mathbb{C}^\times)^n$. We can rephrase Theorem 5.1 as follows.

Theorem 5.3 (cf. [31], Theorem 1.2)**.** *Let V be a subvariety of $\mathbb{C}^n \times (\mathbb{C}^\times)^n$ and let U be an irreducible analytic component of $V \cap D_n$. Assume that the projection of U to $(\mathbb{C}^\times)^n$ is not contained in a coset of a proper subtorus of $(\mathbb{C}^\times)^n$. Then*

$$\dim V \geq \dim U + n.$$

Similarly, we can rephrase Theorem 5.2 as follows.

Theorem 5.4. *Let W be a subvariety of $\mathbb{C}^n$ and V a subvariety of $(\mathbb{C}^\times)^n$. Let A be an irreducible analytic component of $W \cap \pi^{-1}(V)$. If A is not contained in $b + L$, for any proper $\mathbb{Q}$-linear subspace L of $\mathbb{C}^n$ and any $b \in \mathbb{C}^n$, then*

$$\dim V + \dim W \geq \dim A + n.$$

Recall that X is naturally endowed with the structure of a hermitian symmetric domain. In particular, it is a complex manifold. We define a (irreducible algebraic) **subvariety** of X as in Appendix B of [20]. In particular, we consider the Harish-Chandra realization of X as a bounded domain in $\mathbb{C}^N$, for some $N \in \mathbb{N}$, and we define an (irreducible algebraic) subvariety of X to be an irreducible analytic component of the intersection of X with an algebraic subvariety of $\mathbb{C}^N$. We define a (irreducible algebraic) subvariety of $X \times S$ to be an irreducible analytic component of the intersection of $X \times S$ with an algebraic subvariety of $\mathbb{C}^N \times S$.

We are, therefore, able to formulate conjectures for Shimura varieties that are analogous to those above. Let π henceforth denote the uniformization map

$$X \to S$$

and let D_S denote the graph of π in $X \times S$. The following conjecture generalizes Conjecture 1.1 of [27].

Conjecture 5.5 (hyperbolic Ax-Schanuel)**.** *Let V be a subvariety of $X \times S$ and let U be an irreducible analytic component of $V \cap D_S$. Assume that the projection of U to S is not contained in a weakly special subvariety of $\mathrm{Sh}_K(G, \mathfrak{X})$ strictly contained in S. Then*

$$\dim V \geq \dim U + \dim S$$

For $S = \mathbb{C}^n$, Conjecture 5.5 and its generalization involving derivatives were obtained in [27]. Mok, Pila, and Tsimerman have very recently announced a proof of Conjecture 5.5 in full.

For applications to the Zilber-Pink conjecture, only the following weaker version will be needed.

Conjecture 5.6 (cf. [18], Conjecture 5.10)**.** *Let W be a subvariety of X and let V be a subvariety of S. Let A be an irreducible analytic component of $W \cap \pi^{-1}(V)$ and assume that $\pi(A)$ is not contained in a weakly special subvariety of $\mathrm{Sh}_K(G, \mathfrak{X})$ strictly contained in S. Then*

$$\dim V + \dim W \geq \dim A + \dim S.$$

Proof that Conjecture 5.5 implies Conjecture 5.6. Consider the situation described in the statement of Conjecture 5.6. Then $Y := W \times V$ is an algebraic subvariety of $X \times S$ and

$$U := \{(a, \pi(a)); a \in A\}$$

is an irreducible analytic component of $Y \cap D_S$. Clearly, the projection of U to S is not contained in a weakly special subvariety of $\mathrm{Sh}_K(G, \mathfrak{X})$ strictly contained in S. Therefore, by Conjecture 5.5,

$$\dim Y \geq \dim U + \dim S$$

and the result follows since $\dim U = \dim A$ and $\dim Y = \dim W + \dim V$. □

In our applications, we will use a reformulation of Conjecture 5.6. For this reformulation, we will need the following definitions.

Fix a subvariety V of S.

Definition 5.7. An **intersection component** of $\pi^{-1}(V)$ is an irreducible analytic component of the intersection of $\pi^{-1}(V)$ with a subvariety of X.

Clearly, for any intersection component A of $\pi^{-1}(V)$, there exists a smallest subvariety $\langle A \rangle_{\mathrm{Zar}}$ of X containing A.

Definition 5.8. Let A be an intersection component of $\pi^{-1}(V)$. We define the **Zariski defect** of A to be

$$\delta_{\mathrm{Zar}}(A) := \dim \langle A \rangle_{\mathrm{Zar}} - \dim A.$$

Definition 5.9. We say that an intersection component A of $\pi^{-1}(V)$ is **Zariski optimal** in $\pi^{-1}(V)$ if, for any intersection component B of $\pi^{-1}(V)$,

$$A \subsetneq B \subseteq \pi^{-1}(V) \implies \delta_{\mathrm{Zar}}(B) > \delta_{\mathrm{Zar}}(A).$$

Remark 5.10. Let A be an intersection component of $\pi^{-1}(V)$. If A is Zariski optimal in $\pi^{-1}(V)$, then A is an irreducible analytic component of

$$\langle A \rangle_{\mathrm{Zar}} \cap \pi^{-1}(V).$$

Definition 5.11. Let $x \in X$ and let X_M denote the $M(\mathbb{R})^+$ conjugacy class of x in X, where $M := \mathrm{MT}(x)$. Write M^{ad} as a product

$$M^{\mathrm{ad}} = M_1 \times M_2$$

of two normal $\mathbb{Q}$-subgroups, either of which may be trivial, thus inducing a splitting

$$X_M = X_1 \times X_2.$$

For any $x_1 \in X_1$ or $x_2 \in X_2$, we obtain a subvariety $\{x_1\} \times X_2$ or $X_1 \times \{x_2\}$ of X. We refer to any subvariety of X taking this form as a **pre-weakly special subvariety** of X. That is, a weakly special subvariety of $\mathrm{Sh}_K(G, \mathfrak{X})$ contained in S is, by definition, the image in S of a pre-weakly special subvariety of X.

Remark 5.12. A pre-weakly special subvariety of X is totally geodesic in X and an irreducible analytic subset of X.

Definition 5.13. An intersection component A of $\pi^{-1}(V)$ is called **pre-weakly special** if it is an irreducible analytic component of

$$\langle A\rangle_{\mathrm{Zar}} \cap \pi^{-1}(V)$$

and $\langle A\rangle_{\mathrm{Zar}}$ is a pre-weakly special subvariety of X.

Conjecture 5.14 (weak hyperbolic Ax-Schanuel)**.** *Let A be an intersection component of $\pi^{-1}(V)$ that is Zariski optimal in $\pi^{-1}(V)$. Then A is pre-weakly special.*

Note that, since A is assumed to be Zariski optimal in $\pi^{-1}(V)$, it is automatically an irreducible analytic component of

$$\langle A\rangle_{\mathrm{Zar}} \cap \pi^{-1}(V).$$

The content of Conjecture 5.14, therefore, is the claim that $\langle A\rangle_{\mathrm{Zar}}$ is a pre-weakly special subvariety of X. Note also that Conjecture 5.14 is a direct generalization of the hyperbolic Ax-Lindemann theorem.

Theorem 5.15 (hyperbolic Ax-Lindemann)**.** *The maximal subvarieties contained in $\pi^{-1}(V)$ are pre-weakly special.*

Proof that Conjecture 5.14 implies Theorem 5.15. The maximal subvarieties contained in $\pi^{-1}(V)$ are precisely the intersection components of $\pi^{-1}(V)$ that are Zariski optimal in $\pi^{-1}(V)$ and whose Zariski defect is zero. □

Lemma 5.16. *Conjecture 5.6 and Conjecture 5.14 are equivalent.*

Proof. See [18], Section 5. □

6 A finiteness result for weakly optimal subvarieties

In this section, we deduce from the weak hyperbolic Ax-Schanuel conjecture a finiteness statement for the weakly optimal subvarieties of a given subvariety V.

Definition 6.1. Let $x \in X$ and let X_M denote the $M(\mathbb{R})^+$ conjugacy class of x in X, where $M := \mathrm{MT}(x)$. Then X_M is a subvariety of X and we refer to any subvariety of X taking this form as a **pre-special subvariety** of X. In particular, a pre-special subvariety of X is a pre-weakly special subvariety of X. If X_M is a point, that is, if M is a torus, we refer to X_M as a **pre-special point** of X. A special subvariety of $\mathrm{Sh}_K(G,\mathfrak{X})$ contained in S is, by definition, the image in S of a pre-special subvariety of X.

Definition 6.2. Let $x \in X$ and let X_M denote the $M(\mathbb{R})^+$ conjugacy class of x in X, where $M := \mathrm{MT}(x)$. Write M^{ad} as a product

$$M^{\mathrm{ad}} = M_1 \times M_2$$

of two normal $\mathbb{Q}$-subgroups, either of which may be trivial, thus inducing a splitting

$$X_M = X_1 \times X_2.$$

For any $x_1 \in X_1$ or $x_2 \in X_2$, we refer to the pre-weakly special subvariety $\{x_1\}\times X_2$ or $X_1\times\{x_2\}$ of X as a **fiber** of the pre-special subvariety X_M of X.

In particular, if we let M_2 be the trivial group, then the points of X_1 are simply the points of X_M, and they are all fibers of X_M.

The main result of this section is the following.

Proposition 6.3 (cf. [18], Proposition 6.6)**.** *Let V be a subvariety of S and assume that the weak hyperbolic Ax-Schanuel conjecture is true for V. Then there exists a finite set Σ of pre-special subvarieties of X such that the following holds.*

Let W be a subvariety of V that is weakly optimal in V. Then there exists $Y \in \Sigma$ such that $\langle W\rangle_{\mathrm{ws}}$ is equal to the image in S of a fiber of Y.

Note that similar theorems also hold for abelian varieties (see [18], Proposition 6.1 and [29], Proposition 3.2).

In order to prove Proposition 6.3, we will need to introduce some new terminology.

Fix a subvariety V of S.

Lemma 6.4. *Let A be an intersection component of $\pi^{-1}(V)$. There is a smallest subvariety $\langle A\rangle_{\mathrm{geo}}$ of X that is totally geodesic in X and contains A.*

Proof. Suppose that A is contained in two subvarieties T_1 and T_2 of X that are both totally geodesic in X. By Lemma 2.4, $T_1 \cap T_2$ is totally geodesic. It is also a finite union of subvarieties of X, one of which must contain A. The result follows. □

Definition 6.5. Let A be an intersection component of $\pi^{-1}(V)$. We define the **geodesic defect** of A to be

$$\delta_{\mathrm{geo}}(A) := \dim\langle A\rangle_{\mathrm{geo}} - \dim A.$$

We note that, in [18], this notion was referred to as the Möbius defect of A.

Definition 6.6. We say that an intersection component A of $\pi^{-1}(V)$ is **geodesically optimal** in $\pi^{-1}(V)$ if, for any intersection component B of $\pi^{-1}(V)$,

$$A \subsetneq B \subseteq \pi^{-1}(V) \implies \delta_{\mathrm{geo}}(B) > \delta_{\mathrm{geo}}(A).$$

We note that the terminology geodesically optimal has a different meaning in [18].

Remark 6.7. Let A be an intersection component of $\pi^{-1}(V)$. If A is geodesically optimal in $\pi^{-1}(V)$, then A is an irreducible analytic component of

$$\langle A\rangle_{\mathrm{geo}} \cap \pi^{-1}(V).$$

Lemma 6.8. *Assume that the weak hyperbolic Ax-Schanuel conjecture is true for V and let A be an intersection component of $\pi^{-1}(V)$. If A is geodesically optimal in $\pi^{-1}(V)$, then A is Zariski optimal in $\pi^{-1}(V)$.*

Proof. Suppose that B is an intersection component of $\pi^{-1}(V)$ containing A such that

$$\delta_{\mathrm{Zar}}(B) \leq \delta_{\mathrm{Zar}}(A).$$

We can and do assume that B is Zariski optimal and so, by the weak hyperbolic Ax-Schanuel conjecture, B is pre-weakly special. Therefore, $\langle B\rangle_{\mathrm{Zar}}$ is a pre-weakly special subvariety of X and, therefore, equal to $\langle B\rangle_{\mathrm{geo}}$. Then

$$\delta_{\mathrm{geo}}(B) = \delta_{\mathrm{Zar}}(B) \leq \delta_{\mathrm{Zar}}(A) \leq \delta_{\mathrm{geo}}(A),$$

and, since A is geodesically optimal in $\pi^{-1}(V)$, we conclude that $B = A$. □

Remark 6.9. Let W be a subvariety of S and let A denote an irreducible analytic component of $\pi^{-1}(W)$ in X. Then every irreducible analytic component of $\pi^{-1}(W)$ is equal to a Γ translate of A. In particular, $\pi(A)$ is equal to W.

Lemma 6.10. *Assume that the weak hyperbolic Ax-Schanuel conjecture is true for V and let W be a subvariety of V that is weakly optimal in V. Let A be an irreducible analytic component of $\pi^{-1}(W)$. Then A is an intersection component of $\pi^{-1}(V)$ and is geodesically optimal in $\pi^{-1}(V)$.*

Proof. Clearly, A is an intersection component of $\pi^{-1}(V)$ since W is an irreducible component of

$$\langle W\rangle_{\mathrm{ws}} \cap V$$

and $\pi^{-1}\langle W\rangle_{\mathrm{ws}}$ is equal to the Γ orbit of a pre-weakly special subvariety of X.

Therefore, let B be an intersection component of $\pi^{-1}(V)$ containing A such that

$$\delta_{\mathrm{geo}}(B) \leq \delta_{\mathrm{geo}}(A).$$

We can and do assume that B is geodesically optimal in $\pi^{-1}(V)$ and so, by Lemma 6.8, B is Zariski optimal in $\pi^{-1}(V)$. Therefore, by the weak hyperbolic Ax-Schanuel conjecture, B is pre-weakly special i.e. $\langle B\rangle_{\mathrm{Zar}}$ is a pre-weakly special subvariety of X.

Let Z denote the Zariski closure of $\pi(B)$. We claim that $\langle Z\rangle_{\mathrm{ws}} = \pi(\langle B\rangle_{\mathrm{Zar}})$. To see this, note that Z is contained in $\pi(\langle B\rangle_{\mathrm{Zar}})$ and so $\langle Z\rangle_{\mathrm{ws}}$ is contained in $\pi(\langle B\rangle_{\mathrm{Zar}})$. On the other hand, $\langle B\rangle_{\mathrm{Zar}}$ is contained in $\pi^{-1}(\langle Z\rangle_{\mathrm{ws}})$ and so $\pi(\langle B\rangle_{\mathrm{Zar}})$ is contained in $\langle Z\rangle_{\mathrm{ws}}$, which proves the claim. Therefore,

$$\begin{aligned}
\delta_{\mathrm{ws}}(Z) &= \dim \pi(\langle B\rangle_{\mathrm{Zar}}) - \dim Z\\
&= \dim\langle B\rangle_{\mathrm{Zar}} - \dim Z\\
&\leq \dim\langle B\rangle_{\mathrm{Zar}} - \dim B = \delta_{\mathrm{geo}}(B) \leq \delta_{\mathrm{geo}}(A)\\
&\leq \dim\langle W\rangle_{\mathrm{ws}} - \dim W = \delta_{\mathrm{ws}}(W).
\end{aligned}$$

Since W is weakly optimal in V and contained in Z, we conclude that Z is equal to W. In particular, B is contained in $\pi^{-1}(W)$ and, therefore, equal to A. $\square$

As explained in [20], we can and do fix, once and for all, an open, semialgebraic fundamental set $\mathcal{F}$ in X for the action of Γ such that the set $\mathcal{V} := \pi^{-1}(V) \cap \mathcal{F}$ is definable. The key step in the proof of Proposition 6.3 is the following.

Proposition 6.11. *Assume that the weak hyperbolic Ax-Schanuel theorem is true for V. There exists a finite set Σ of pre-special subvarieties of X such that the following holds.*

Let A be an intersection component of $\pi^{-1}(V)$ that is pre-weakly special such that, for some $x \in \langle A\rangle_{\mathrm{Zar}}$,

$$\dim A = \dim_x(\langle A\rangle_{\mathrm{Zar}} \cap \mathcal{V}).$$

Then there exists $Y \in \Sigma$ such that $\langle A\rangle_{\mathrm{Zar}}$ is equal to a fiber of Y.

In order to prove Proposition 6.11, we require some further preparations.

Definition 6.12. We say that a real semisimple algebraic group F is **without compact factors** if it is equal to an almost direct product of almost simple subgroups whose underlying real Lie groups are not compact. We allow the product to be trivial i.e. we consider the trivial group as a real semisimple algebraic group without compact factors.

Lemma 6.13. *A subvariety of X that is totally geodesic in X is of the form*

$$F(\mathbb{R})^+ \cdot x,$$

where F is a semisimple algebraic subgroup of $G_{\mathbb{R}}$ without compact factors and $x \in X$ factors through

$$G_F := F \cdot Z_{G_{\mathbb{R}}}(F)^\circ.$$

Conversely, if F is a semisimple algebraic subgroup of $G_{\mathbb{R}}$ without compact factors and $x \in X$ factors through G_F, then $F(\mathbb{R})^+x$ is a subvariety of X that is totally geodesic in X.

Proof. See [34], Proposition 2.3. □

We let Ω denote a (finite) set of representatives for the $G(\mathbb{R})$-conjugacy classes of semisimple algebraic subgroups of $G_{\mathbb{R}}$ that are without compact factors. It is clear that the set

$$\Pi_0 := \{(x, g, F) \in \mathcal{V} \times G(\mathbb{R}) \times \Omega : x(\mathbb{S}) \subseteq gG_F g^{-1}\},$$

parameterising (albeit in a many-to-one fashion) the totally geodesic subvarieties of X passing through $\mathcal{V}$, is definable.

Recall from [38], 1.17 that the **local dimension** $\dim_x A$ of a definable set A at a point $x \in A$ is definable. By [18], Lemma 6.2, if A is also a (complex) analytic set, then this dimension is exactly twice the local analytic dimension at x. Furthermore, if A is analytically irreducible, then its local dimension at the points of A is constant. For the remainder of this section, dimensions will be taken in the sense of definable sets.

Consider the two functions

$$\begin{aligned} d(x, g, F) &:= \dim_x(gF(\mathbb{R})^+g^{-1}x) = \dim(gF(\mathbb{R})^+g^{-1}x) \\ d_{\mathcal{V}}(x, g, F) &:= \dim_x(\mathcal{V} \cap gF(\mathbb{R})^+g^{-1}x), \end{aligned}$$

and let Π_1 denote the definable set

$$\begin{aligned} \{(x, g, F) \in \Pi_0 : (x, g_1, F_1) \in \Pi_0,\ gF(\mathbb{R})^+g^{-1} \cdot x \subsetneq g_1F_1(\mathbb{R})^+g_1^{-1} \cdot x \\ \implies d(x, g, F) - d_{\mathcal{V}}(x, g, F) < d(x, g_1, F_1) - d_{\mathcal{V}}(x, g_1, F_1)\}. \end{aligned}$$

Finally, let Π_2 denote the definable set

$$\begin{aligned} \{(x, g, F) \in \Pi_1 : (x, g_1, F_1) \in \Pi_0,\ g_1F'(\mathbb{R})^+g_1^{-1} \cdot x \subsetneq gF(\mathbb{R})^+g^{-1} \cdot x \\ \implies d_{\mathcal{V}}(x, g_1, F_1) < d_{\mathcal{V}}(x, g, F)\}. \end{aligned}$$

The proof of Proposition 6.11 will require the following three lemmas.

Lemma 6.14. *Let A be an intersection component of $\pi^{-1}(V)$ that is pre-weakly special such that, for some $x \in \langle A\rangle_{\mathrm{Zar}}$,*

$$\dim A = \dim_x(\langle A\rangle_{\mathrm{Zar}} \cap \mathcal{V}).$$

Then we can write

$$\langle A\rangle_{\mathrm{Zar}} = gF(\mathbb{R})^+g^{-1} \cdot x,$$

where $(x, g, F) \in \Pi_2$.

Proof. By Lemma 6.13, we can write

$$\langle A\rangle_{\mathrm{Zar}} = gF(\mathbb{R})^+g^{-1}\cdot x$$

for some $F\in\Omega$ and some $x\in\mathcal{V}$ that factors through gG_Fg^{-1}. In particular, $(x,g,F)\in\Pi_0$. By assumption, we can and do choose $x\in\langle A\rangle_{\mathrm{Zar}}$ such that

$$\dim A = \dim_x(\langle A\rangle_{\mathrm{Zar}}\cap\mathcal{V}) = d_{\mathcal{V}}(x,g,F).$$

Suppose that (x,g,F) does not belong to Π_1 i.e. that there exists $(x,g_1,F_1)\in\Pi_0$ such that

$$gF(\mathbb{R})^+g^{-1}\cdot x\subsetneq g_1F_1(\mathbb{R})^+g_1^{-1}\cdot x,$$

and

$$d(x,g,F)-d_{\mathcal{V}}(x,g,F)\geq d(x,g_1,F_1)-d_{\mathcal{V}}(x,g_1,F_1). \tag{6.14.1}$$

Let B be an irreducible analytic component of

$$g_1F_1(\mathbb{R})^+g_1^{-1}\cdot x\cap\pi^{-1}(V)$$

passing through x such that

$$\dim B = d_{\mathcal{V}}(x,g_1,F_1).$$

From (6.14.1), we obtain $\delta_{\mathrm{Zar}}(B)\leq\delta_{\mathrm{Zar}}(A)$.

On the other hand, the Intersection Inequality (see [15], Chapter 5, §3) yields

$$\dim_x(B\cap\langle A\rangle_{\mathrm{Zar}})\geq\dim B+\dim(\langle A\rangle_{\mathrm{Zar}})-d(x,g_1,F_1)$$

and, from (6.14.1), we obtain

$$\dim_x(B\cap\langle A\rangle_{\mathrm{Zar}})\geq\dim(A).$$

It follows that $B\cap\langle A\rangle_{\mathrm{Zar}}$, and hence B itself, contains a complex neighbourhood of x in A, which implies that A is contained in B.

Therefore, since A is Zariski optimal, we conclude that $A=B$. However, this implies that

$$d(x,g_1,F_1)-d_{\mathcal{V}}(x,g_1,F_1)>2\delta_{\mathrm{Zar}}(B)=2\delta_{\mathrm{Zar}}(A)=d(x,g,F)-d_{\mathcal{V}}(x,g,F),$$

which contradicts (6.14.1).

Therefore, suppose that $(x,g,F)\in\Pi_1$ does not belong to Π_2 i.e. that there exists $(x,g_1,F_1)\in\Pi_0$ such that

$$g_1F_1(\mathbb{R})^+g_1^{-1}\cdot x\subsetneq gF(\mathbb{R})^+g^{-1}\cdot x,$$

and

$$d_{\mathcal{V}}(x,g_1,F_1)=d_{\mathcal{V}}(x,g,F)=\dim A.$$

But then A is contained in

$$g_1F_1(\mathbb{R})^+g_1^{-1}\cdot x\subsetneq\langle A\rangle_{\mathrm{Zar}},$$

which is contradiction. $\square$

Lemma 6.15. *Assume that the weak hyperbolic Ax-Schanuel conjecture is true for V. Then, if $(x, g, F) \in \Pi_2$, there exists a semisimple subgroup F' of G defined over $\mathbb{Q}$ such that gFg^{-1} is equal to the almost direct product of the almost simple factors of $F'_{\mathbb{R}}$ whose underlying real Lie groups are non-compact.*

Proof. By [33], Proposition 3.1, it suffices to show that $gF(\mathbb{R})^+g^{-1} \cdot x$ is a pre-weakly special subvariety of X. Therefore, let A be an irreducible analytic component of

$$gF(\mathbb{R})^+g^{-1} \cdot x \cap \pi^{-1}(V)$$

passing through x such that

$$\dim A = d_{\mathcal{V}}(x, g, F).$$

Let B be an intersection component of $\pi^{-1}(V)$ containing A such that $\delta_{\mathrm{Zar}}(B) \leq \delta_{\mathrm{Zar}}(A)$. We can and do assume that B is Zariski optimal and, therefore, by the weak hyperbolic Ax-Schanuel conjecture, pre-weakly special i.e. B is an irreducible component of

$$\langle B \rangle_{\mathrm{Zar}} \cap \pi^{-1}(V)$$

and $\langle B \rangle_{\mathrm{Zar}}$ is a pre-weakly special subvariety of X. Therefore, A is contained in

$$gF(\mathbb{R})^+g^{-1} \cdot x \cap \langle B \rangle_{\mathrm{Zar}}$$

and so, since $(x, g, F) \in \Pi_2$, we conclude that $gF(\mathbb{R})^+g^{-1} \cdot x$ is contained in $\langle B \rangle_{\mathrm{Zar}}$.

We also have

$$\begin{aligned}\dim\langle B \rangle_{\mathrm{Zar}} - \dim_x(\langle B \rangle_{\mathrm{Zar}} \cap \mathcal{V}) &\leq \delta_{\mathrm{Zar}}(B) \leq \delta_{\mathrm{Zar}}(A)\\ &\leq d(x, g, F) - \dim A\\ &= d(x, g, F) - d_{\mathcal{V}}(x, g, F),\end{aligned}$$

and so, since $(x, g, F) \in \Pi_1$, we conclude that

$$gF(\mathbb{R})^+g^{-1} \cdot x = \langle B \rangle_{\mathrm{Zar}}.$$

$\square$

Lemma 6.16. *Assume that the weak hyperbolic Ax-Schanuel conjecture is true for V. Then, the set*

$$\{gFg^{-1} : (x, g, F) \in \Pi_2\}$$

is finite.

Proof. Decompose Π_2 as the finite union of the Π_F, where F varies over the members of Ω and Π_F denotes the set of tuples $(x, g, F) \in \Pi_2$. For each $F \in \Omega$, consider the map

$$\Pi_F \to G(\mathbb{R})/N_{G(\mathbb{R})}(F),$$

defined by

$$(x, g, F) \mapsto gN_{G(\mathbb{R})}(F),$$

whose image is, therefore, a definable set. However, Lemma 6.15, implies that it is countable and hence finite. $\square$

Proof of Proposition 6.11. Let A be an intersection component of $\pi^{-1}(V)$ that is pre-weakly special such that, for some $x \in \langle A\rangle_{\mathrm{Zar}}$,

$$\dim A = \dim_x(\langle A\rangle_{\mathrm{Zar}} \cap \mathcal{V}).$$

Then, by Lemma 6.14, we can write

$$\langle A\rangle_{\mathrm{Zar}} = gF(\mathbb{R})^+g^{-1} \cdot x$$

where $(x, g, F) \in \Pi_2$. By Lemma 6.15, there exists a semisimple subgroup F' of G defined over $\mathbb{Q}$ such that gFg^{-1} is equal to the almost direct product of the almost simple factors of $F'_{\mathbb{R}}$ whose underlying real Lie groups are non-compact. In fact, by [33], Proposition 3.1, F' is smallest subgroup of G defined over $\mathbb{Q}$ containing gFg^{-1}. Since, by Lemma 6.16, gFg^{-1} comes from a finite set, so too does F'. In particular, the reductive algebraic group

$$M := F' \cdot Z_G(F')^\circ$$

is defined over $\mathbb{Q}$ and belongs to a finite set.

If we write M^{nc} for the almost direct product of the almost $\mathbb{Q}$-simple factors of M whose underlying real Lie groups are not compact, then x factors through $M'_{\mathbb{R}} := Z(M)^\circ_{\mathbb{R}} \cdot M^{\mathrm{nc}}_{\mathbb{R}}$ and, if we write $\mathfrak{X}_M$ for the $M'(\mathbb{R})$ conjugacy class of x in $\mathfrak{X}$, then, by [32], Lemme 3.3, $(M', \mathfrak{X}_M)$ is a Shimura subdatum of $(G, \mathfrak{X})$. Furthermore, by [36], Lemma 3.7, the number of Shimura subdatum $(M', \mathfrak{Y})$ is finite. Therefore, since the $M'(\mathbb{R})^+$ conjugacy class X_M of x in X is a pre-special subvariety of X and $\langle A\rangle_{\mathrm{Zar}}$ is a fiber of X_M, the proof is complete. □

Proof of Proposition 6.3. Let A be an irreducible analytic component of $\pi^{-1}(W)$. By Lemma 6.10, A is an intersection component of $\pi^{-1}(V)$ and is geodesically optimal in $\pi^{-1}(V)$. Therefore, by Lemma 6.8, A is Zariski optimal in $\pi^{-1}(V)$ and so, by the weak hyperbolic Ax-Schanuel conjecture, A is pre-weakly special. It follows that the image of $\langle A\rangle_{\mathrm{Zar}}$ in S is equal to $\langle W\rangle_{\mathrm{ws}}$.

After possibly replacing A by a γA, for some $\gamma \in \Gamma$, we can and do assume that there exists $x \in \langle A\rangle_{\mathrm{Zar}}$ such that

$$\dim(A) = \dim_x(\langle A\rangle_{\mathrm{Zar}} \cap \mathcal{V}).$$

By Proposition 6.11, $\langle A\rangle_{\mathrm{Zar}}$ is a fiber of $Y \in \Sigma$, where Σ is a finite set of pre-special subvarieties of X depending only on V. □

7 Anomalous subvarieties

In this section, we recall the notion of an anomalous subvariety, which is defined by Bombieri, Masser, and Zannier in [5] for subvarieties of algebraic tori. In fact, we give the more general notion of an r-anomalous subvariety, as introduced by Rémond [29].

Let V be a subvariety of S. We will use Proposition 6.3 to show that, under the weak hyperbolic Ax-Schanuel conjecture, the union of the subvarieties of V that are r-anomalous in V constitutes a Zariski closed subset of V. We will then give a criterion for when it is a proper subset.

Definition 7.1. Let $r \in \mathbb{Z}$. A subvariety W of V is called **r-anomalous** in V if

$$\dim W \geq \max\{1, r + \dim\langle W\rangle_{\mathrm{ws}} - \dim S\}.$$

A subvariety of V is **maximal r-anomalous** in V if it is r-anomalous in V and not strictly contained in another subvariety of V that is also r-anomalous in V.

We denote by $\mathrm{an}(V,r)$ the set of subvarieties of V that are r-anomalous in V and by $V^{\mathrm{an},r}$ the union of the elements of $\mathrm{an}(V,r)$.

We say that a subvariety of V is **anomalous** if it is $(1+\dim V)$-anomalous. We write $\mathrm{an}(V)$ for $\mathrm{an}(V,1+\dim V)$ and V^{an} for $V^{\mathrm{an},1+\dim V}$.

Theorem 7.2. *Assume that the weak hyperbolic Ax-Schanuel conjecture is true for V and let $r\in\mathbb{Z}$. Then $V^{\mathrm{an},r}$ is a Zariski closed subset of V.*

We refer the reader to [5], [29], and [18] for similar results on algebraic tori and abelian varieties. We will require the following facts.

Proposition 7.3 (cf. [19], Chapter 2, Exercise 3.22 (d))**.** *Let $f:W\to Y$ be a dominant morphism between two integral schemes of finite type over a field and let*

$$e:=\dim W-\dim Y$$

denote the **relative dimension**. *For $h\in\mathbb{Z}$, let E_h denote the set of points $x\in W$ such that the fibre $f^{-1}(f(x))$ possesses an irreducible component of dimension at least h that contains x. Then*

(1) E_h is a Zariski closed subset of W,

(2) $E_e=W$, and

(3) if $h>e$, E_h is not Zariski dense in W.

Lemma 7.4. *Let $W\in\mathrm{an}(V,r)$. Then W is weakly optimal in V.*

Proof. Let Y be a subvariety of V containing W such that $\delta_{\mathrm{ws}}(Y)\le\delta_{\mathrm{ws}}(W)$. Then

$$\begin{aligned}\dim Y&=\dim\langle Y\rangle_{\mathrm{ws}}-\delta_{\mathrm{ws}}(Y)\\&\ge\dim\langle Y\rangle_{\mathrm{ws}}-\delta_{\mathrm{ws}}(W)\\&=\dim\langle Y\rangle_{\mathrm{ws}}-(\dim\langle W\rangle_{\mathrm{ws}}-\dim W)\\&\ge\dim\langle Y\rangle_{\mathrm{ws}}+r-\dim S.\end{aligned}$$

Since Y contains W, we know that $\dim Y\ge 1$ and so $Y\in\mathrm{an}(V,r)$. We conclude that Y must be equal to W and, therefore, that W is weakly optimal in V. □

Proof of Theorem 7.2. Let Σ be a finite set of pre-special subvarieties of X (whose existence is ensured by Proposition 6.3) such that, if W is a subvariety of V that is weakly optimal in V, then there exists $x\in X$ such that, if $M:=\mathrm{MT}(x)$, then the $M(\mathbb{R})^+$ conjugacy class X_M of x in X belongs to Σ and $\langle W\rangle_{\mathrm{ws}}$ is equal to the image in S of a fiber of X_M. That is, we may write M^{ad} as a product $M_1\times M_2$ of two normal $\mathbb{Q}$-subgroups, which induces a splitting $X=X_1\times X_2$ such that $\langle W\rangle_{\mathrm{ws}}$ is equal to the image in S of $\{x_1\}\times X_2$, for some $x_1\in X_1$.

Let $W\in\mathrm{an}(V,r)$. By Lemma 7.4, there exists $X_M\in\Sigma$ such that $\langle W\rangle_{\mathrm{ws}}$ is equal to the image in S of $\{x_1\}\times X_2$, for some $x_1\in X_1$, where $X_M=X_1\times X_2$, as above.

Let Γ_M be a congruence subgroup of $M(\mathbb{Q})_+$ contained in Γ, where $M(\mathbb{Q})_+$ denotes the subgroup of $M(\mathbb{Q})$ acting on X_M, and let Γ_1 denote the image of Γ under the natural maps

$$M(\mathbb{Q})\to M^{\mathrm{ad}}(\mathbb{Q})\to M_1(\mathbb{Q}).$$

We denote by f the restriction of

$$\Gamma_M\backslash X_M\to\Gamma_1\backslash X_1$$

to $\phi^{-1}(V)$, where ϕ denotes the natural map

$$\Gamma_M\backslash X_M \to \Gamma\backslash X = S.$$

Therefore, by Proposition 7.3 (1), the set E_f of points z in $\phi^{-1}(V)$ such that the fibre $f^{-1}(f(z))$ possesses an irreducible component of dimension at least h that contains z is a Zariski closed subset of $\phi^{-1}(V)$. Since ϕ is a closed morphism, $\phi(E_f)$ is Zariski closed in V.

We claim that W is contained in $\phi(E_f)$, where

$$h := \max\{1, r + \dim X_2 - \dim S\},$$

which is equivalent to the claim that any irreducible component of $\phi^{-1}(W)$ is contained in E_f.

Fix such a component W_M. Then $\langle W_M\rangle_{\rm ws}$ is equal to the image of $\{x_1\} \times X_2$ in $\Gamma_M\backslash X_M$ and so W_M lies in a fiber of f. Since

$$\dim W_M = \dim W \geq \max\{1, r + \dim\langle W\rangle_{\rm ws} - \dim S\} = \max\{1, r + \dim X_2 - \dim S\},$$

our claim follows.

Finally, we claim that $\phi(E_f)$ is contained in $V^{{\rm an},r}$. To see this, let $z \in E_h$ and let Y be an irreducible component of the fibre $f^{-1}(f(z))$ of dimension at least h containing z. Then Y is contained in the image of $\{x_1\} \times X_2$ in $\Gamma_M\backslash X_M$, where $x_1 \in X_1$ lies above $f(z) \in \Gamma_1\backslash X_1$, and so

$$\dim\langle Y\rangle_{\rm ws} \leq \dim X_2.$$

Therefore,

$$\dim \phi(Y) = \dim Y \geq h = \max\{1, r + \dim X_2 - \dim S\} \geq \max\{1, r + \dim\langle\phi(Y)\rangle_{\rm ws} - \dim S\}$$

and so $\phi(Y) \in {\rm an}(V, r)$.

Hence, if we let E denote the union of the $\phi(E_f)$ as we vary over the finitely many maps f obtained from the $X_M \in \Sigma$ and their possible splittings, we conclude that $E = V^{{\rm an},r}$, which finishes the proof. □

We denote by $V^{\rm oa}$ the complement in V of $V^{\rm an}$. By Theorem 7.2, this is a (possibly empty) open subset of V. In the literature, it is sometimes referred to as the open-anomalous locus, hence the subscript. We conclude this section by showing that, when V is sufficiently generic, $V^{\rm oa}$ is not empty.

Proposition 7.5. *Suppose that V is Hodge generic in S. Then $V^{\rm an} = V$ if and only if we can write $G^{\rm ad} = G_1 \times G_2$, and thus $X = X_1 \times X_2$, such that*

$$\dim f(V) < \min\{\dim V, \dim X_1\},$$

where f denotes the projection map

$$\Gamma\backslash X \to \Gamma_1\backslash X_1,$$

and Γ_1 denotes the image of Γ under the natural maps

$$G(\mathbb{Q}) \to G^{\rm ad}(\mathbb{Q}) \to G_1(\mathbb{Q}).$$

Proof. First suppose that $V^{\mathrm{an}} = V$. Then, for any set Σ as in the proof of Theorem 7.2, V is contained in the (finite) union of the images in X of the $X_M \in \Sigma$. Therefore, since V is assumed to be Hodge generic in S, it must be that $X \in \Sigma$ and, furthermore, that there exists $W \in \mathrm{an}(V)$ such that $G^{\mathrm{ad}} = G_1 \times G_2$, and thus $X = X_1 \times X_2$, such that $\langle W \rangle_{\mathrm{ws}}$ is equal to the image in S of $\{x_1\} \times X_2$, for some $x_1 \in X_1$.

Let f denote the projection map

$$\Gamma \backslash X \to \Gamma_1 \backslash X_1,$$

as in the statement of the proposition, and consider its restriction

$$V \to \overline{f(V)},$$

where $\overline{f(V)}$ denotes the Zariski closure of $f(V)$ in $\Gamma_1 \backslash X_1$. Since $V^{\mathrm{an}} = V$, it follows from Proposition 7.3 (3), that

$$h := \max\{1, 1 + \dim V + \dim X_2 - \dim X\} \leq \dim V - \dim f(V).$$

Hence,

$$\dim f(V) < \dim X - \dim X_2 = \dim X_1.$$

Furthermore, since V contains W, which is of positive dimension and contained in the image in S of $\{x_1\} \times X_2$, we conclude that the dimension of $f(V)$ is strictly smaller than the dimension of V.

Conversely, suppose that $G^{\mathrm{ad}} = G_1 \times G_2$, and thus $X = X_1 \times X_2$, such that

$$\dim f(V) < \min\{\dim V, \dim X_1\},$$

where f denotes the projection map

$$\Gamma \backslash X \to \Gamma_1 \backslash X_1,$$

as above. Restricting f to

$$V \to \overline{f(V)},$$

as before, we see from Proposition 7.3 (2) that the set E_f of points z in V such that the fibre $f^{-1}(f(z))$ possesses an irreducible component of dimension at least

$$h := \max\{1, 1 + \dim V - \dim X_1\} \leq \dim V - \dim f(V) = \dim V - \dim \overline{f(V)}$$

that contains z is equal to V. However, from the proof of Theorem 7.2, we have seen that E_f is contained in V^{an}, so the claim follows. □

Corollary 7.6. *If G^{ad} is $\mathbb{Q}$-simple and V is a Hodge generic subvariety in S, then V^{an} is strictly contained in V. In particular, V^{an} is strictly contained in V whenever V is a Hodge generic subvariety of $\mathcal{A}_g$.*

8 Main results (part 1): Reductions to point counting

In this section, we prove our main theorem, that, under the weak hyperbolic Ax-Schanuel conjecture, the Zilber-Pink conjecture can be reduced to a problem of point counting. We also give a reduction of Pink's conjecture in the case when the open-anomalous locus is non-empty.

Definition 8.1. Let V be a subvariety of S. We denote by $\mathrm{Opt}_0(V)$ the set of all points in V that are optimal in V.

Consider the following corollary of the Zilber-Pink conjecture.

Conjecture 8.2. *Let V be a subvariety of S. Then $\mathrm{Opt}_0(V)$ is finite.*

We will later show that, under certain arithemtic hypotheses, one can prove Conjecture 8.2 when V is a curve. Our main result in this section is that (under the weak hyperbolic Ax-Schanuel conjecture), Conjecture 8.2 implies the Zilber-Pink conjecture.

Theorem 8.3. *Assume that the weak hyperbolic Ax-Schanuel conjecture is true and assume that Conjecture 8.2 holds.*

Let V be a subvariety of S. Then $\mathrm{Opt}(V)$ is finite.

Proof. We prove Theorem 8.3 by induction on $\dim V$. Of course, Theorem 8.3 is trivial when $\dim V = 0$ or $\dim V = 1$. Therefore, we assume that $\dim V \geq 2$ and that Theorem 8.3 holds whenever the subvariety in question is of lower dimension.

We need to show that the induction hypothesis implies that there are only finitely many subvarieties of positive dimension belonging to $\mathrm{Opt}(V)$.

Let Σ be a finite set of pre-special subvarieties of X, as in the proof of Theorem 7.2, and let $W \in \mathrm{Opt}(V)$ be of positive dimension.

By Corollary 4.5, W is weakly-optimal and, therefore, there exists $x \in X$ such that, if $M := \mathrm{MT}(x)$, the $M(\mathbb{R})^+$ conjugacy class X_M of x in X belongs to Σ and $\langle W\rangle_{\mathrm{ws}}$ is equal to the image in S of a fiber of X_M. That is, we may write M^{ad} as a product

$$M^{\mathrm{ad}} = M_1 \times M_2$$

of two normal $\mathbb{Q}$-subgroups, thus inducing a splitting

$$X_M = X_1 \times X_2,$$

such that $\langle W\rangle_{\mathrm{ws}}$ is equal to the image in S of $\{x_1\} \times X_2$, for some $x_1 \in X_1$.

Let Γ_M be a congruence subgroup of $M(\mathbb{Q})_+$ contained in Γ, where $M(\mathbb{Q})_+$ denotes the subgroup of $M(\mathbb{Q})$ acting on X_M, such that the image of Γ_M under the natural map

$$M(\mathbb{Q}) \to M^{\mathrm{ad}}(\mathbb{Q}) = M_1(\mathbb{Q}) \times M_2(\mathbb{Q})$$

is equal to a product $\Gamma_1 \times \Gamma_2$. We denote by f the natural morphism

$$\Gamma_M\backslash X_M \to \Gamma_1\backslash X_1,$$

and by ϕ the closed morphism

$$\Gamma_M\backslash X_M \to \Gamma\backslash X = S.$$

Let $\widetilde{V}$ be an irreducible component of $\phi^{-1}(V)$ and let $\widetilde{W}$ denote an irreducible component of $\phi^{-1}(W)$. Then $\widetilde{W}$ is optimal in $\widetilde{V}$. On the other hand, by the generic smoothness property, there exists a dense open subset V_0 of $\widetilde{V}$ such that the restriction f_0 of f to V_0 is a smooth morphism of relative dimension ν. We denote by V_1 the Zariski clsoure of $f(V_0)$ in $\Gamma_1\backslash X_1$.

Now suppose that

$$\widetilde{W} \cap V_0 = \emptyset. \tag{8.3.1}$$

Then $\widetilde{W}$ is a subvariety of some irreducible component V^0 of $\widetilde{V} \setminus V_0$. Furthermore, $\widetilde{W}$ is optimal in V^0. However, since $\dim V^0$ is strictly less than $\dim V$, our induction hypothesis implies that $\mathrm{Opt}(V^0)$ is finite.

Therefore, we assume that (8.3.1) does not hold. As an irreducible component of the fibre $f_0^{-1}(z)$, where z denotes the image of x_1 in V_1, its dimension is equal to ν. In particular,

$$\dim \widetilde{W} = \nu.$$

We claim that z is optimal in V_1. To see this, note that $f(\langle \widetilde{W} \rangle)$ contains z and is a special subvariety of dimension

$$\dim \langle \widetilde{W} \rangle - \dim X_2 = \dim \widetilde{W} + \delta(\widetilde{W}) - \dim X_2 = \nu + \delta(\widetilde{W}) - \dim X_2.$$

Therefore, let A be a subvariety of V_1 containing z such that

$$\delta(A) \leq \delta(z) \leq \nu + \delta(\widetilde{W}) - \dim X_2,$$

and let B be an irreducible component of $f^{-1}(A)$ containing $\widetilde{W}$. Since V_0 is open in $\widetilde{V}$ and A is contained in V_1,

$$\dim B = \dim A + \nu.$$

Therefore,

$$\delta(B) \leq \dim\langle A \rangle + \dim X_2 - \dim B = \delta(A) + \dim A + \dim X_2 - \dim B \leq \delta(\widetilde{W})$$

and, since $\widetilde{W}$ is optimal in $\widetilde{V}$, we conclude that B is equal to $\widetilde{W}$. In particular, $\widetilde{W}$ is an irreducible component of $f^{-1}(A)$ but, since it is also contained in $f^{-1}(z)$, it must be that A was z, proving the claim.

Since W was assumed to be of positive dimension, so too must be X_2. It follows that $\dim V_1$ is strictly less than $\dim V$ and so, by the induction hypothesis, $\mathrm{Opt}(V_1)$ is finite. Since $z \in \mathrm{Opt}(V_1)$ and since Σ and the number of splittings are finite, we are done.

□

We will later prove that the following conjecture is a consequence of the weak hyperbolic Ax-Schanuel conjecture and our arithmetic conjectures. It is inspired by the cited theorem of Habegger and Pila.

Conjecture 8.4 (cf. [18], Theorem 9.15 (iii))**.** *Let V be a subvariety of S. Then the set $V^{\mathrm{oa}} \cap S^{[1+\dim V]}$ is finite.*

The importance of Conjecture 8.4 for us is that, when V is suitably generic, Conjecture 8.4 implies Pink's conjecture (assuming the weak hyperbolic Ax-Schanuel conjecture).

Theorem 8.5. *Assume that the weak hyperbolic Ax-Schanuel conjecture is true and that Conjecture 8.4 holds.*

Let V be a Hodge generic subvariety of S such that (even after replacing Γ) S cannot be decomposed as a product $S_1 \times S_2$ such that V is contained in $V' \times S_2$, where V' is a proper subvariety of S_1 of dimension strictly less than the dimension of V. Then

$$V \cap S^{[1+\dim V]}$$

is not Zariski dense in V.

Proof. We claim that the assumptions guarantee that V^{an} is strictly contained in V. Otherwise, by Proposition 7.5, we can write $G^{\mathrm{ad}} = G_1 \times G_2$, and thus $X = X_1 \times X_2$, such that

$$\dim f(V) < \min\{\dim V, \dim X_1\},$$

where f denotes the projection map

$$\Gamma\backslash X \to \Gamma_1\backslash X_1,$$

and Γ_1 denotes the image of Γ under the natural maps

$$G(\mathbb{Q}) \to G^{\mathrm{ad}}(\mathbb{Q}) \to G_1(\mathbb{Q}).$$

Therefore, after replacing Γ, we can write S as a product $S_1 \times S_2$ so that f is simply the projection on to the first factor and V is contained in $V' \times S_2$, where V' is Zariski closure of $f(V)$ in S_1. However, since

$$\dim V' = \dim f(V),$$

this is a contradiction.

Therefore, by Theorem 7.2, V^{an} is a proper Zariski closed subset of V. On the other hand, $V \cap S^{[1+\dim V]}$ is contained in

$$V^{\mathrm{an}} \cup \left[V^{\mathrm{oa}} \cap S^{[1+\dim V]}\right]$$

and so the theorem follows from Conjecture 8.4. □

9 The counting theorem

Henceforth, we turn our attention to the counting problems themselves. We will approach these problems using a theorem of Pila and Wilkie concerned with counting points in definable sets. We first recall the notations.

Let $k \geq 1$ be an integer. For any real number y, we define its k**-height** as

$$\mathrm{H}_k(y) := \min\{\max\{|a_0|, ..., |a_k|\} : a_i \in \mathbb{Z},\ \gcd\{a_0, ..., a_k\} = 1,\ a_0 y^k + ... + a_k = 0\},$$

where we use the convention that, if the set is empty i.e. y is not algebraic of degree at most k, then $\mathrm{H}_k(y)$ is $+\infty$. For $y = (y_1, ..., y_m) \in \mathbb{R}^m$, we set

$$\mathrm{H}_k(y) := \max\{\mathrm{H}_k(y_1), ..., \mathrm{H}_k(y_m)\}.$$

For any set $A \subseteq \mathbb{R}^m \times \mathbb{R}^n$, and for any real number $T \geq 1$, we define

$$A(k, T) := \{(y, z) \in Z : \mathrm{H}_k(y) \leq T\}.$$

The counting theorem of Pila and Wilkie is stated as follows.

Theorem 9.1 (cf. the proof of [18], Corollary 7.2)**.** *Let $D \subseteq \mathbb{R}^l \times \mathbb{R}^m \times \mathbb{R}^n$ be a definable family parametrised by $\mathbb{R}^l$, let k be a positive integer, and let $\epsilon > 0$. There exists a constant $c := c(D, k, \epsilon) > 0$ with the following properties.*

Let $x \in \mathbb{R}^l$ and let

$$D_x := \{(y, z) \in \mathbb{R}^m \times \mathbb{R}^n : (x, y, z) \in D\}.$$

Let π_1 and π_2 denote the projections $\mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$ and $\mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$, respectively. If $T \geq 1$ and $\Sigma \subseteq D_x(k,T)$ satisfies

$$\#\pi_2(\Sigma) > cT^{\epsilon},$$

there exists a continuous and definable function $\beta : [0,1] \to D_x$ such that the following properties hold.

1. *The composition $\pi_1 \circ \beta : [0,1] \to \mathbb{R}^m$ is semi-algebraic.*
2. *The composition $\pi_2 \circ \beta : [0,1] \to \mathbb{R}^n$ is non-constant.*
3. *We have $\beta(0) \in \Sigma$.*

Note that, although the conclusion $\beta(0) \in \Sigma$ does not appear in the statement of [18], Corollary 7.2, it is, indeed, established in its proof.

10 Complexity

In order to apply the counting theorem, we will need a way of counting special points and, more generally, special subvarieties.

Let P be a special point in S and let $x \in X$ be a pre-special point lying above P. In particular, $T := \mathrm{MT}(x)$ is a torus and we denote by D_T the absolute value of the discriminant of its splitting field. We let K_T^m denote the maximal compact open subgroup of $T(\mathbb{A}_f)$ and we let K_T denote $K \cap T(\mathbb{A}_f)$.

Definition 10.1. The **complexity** of P is the natural number

$$\Delta(P) := \max\{D_T, [K_T^m : K_T]\}.$$

Note that this does not depend on the choice of x.

Now let Z be a special subvariety of S. There exists a Shimura subdatum $(H, \mathfrak{X}_H)$ of $(G, \mathfrak{X})$, such that H is the generic Mumford-Tate group on $\mathfrak{X}_H$, and a connected component X_H of $\mathfrak{X}_H$ contained in X such that Z is the image of X_H in $\Gamma\backslash X$. In fact, these choices are well-defined up to conjugation by Γ.

By the **degree** $\deg(Z)$ of Z, we refer to the degree of the Zariski closure of Z in the Baily-Borel compactification of S, defined in [4], which is naturally a projective variety.

Definition 10.2. The **complexity** of Z is the natural number

$$\Delta(Z) := \max\{\deg(Z), \min\{\Delta(P) : P \in Z \text{ is a special point}\}\}.$$

Note that when Z is a special point, this complexity coincides with the former.

This is a natural generalization of the complexities given in [18], Definition 3.4 and Definition 3.8. In order to count special subvarieties, however, it is crucial that the complexity of Z satisfies the following property.

Conjecture 10.3. *For any $b \geq 1$, we have*

$$\#\{Z \subseteq S : Z \text{ is special and } \Delta(Z) \leq b\} < \infty.$$

The obstruction to proving that this property holds for a general Shimura variety can be expressed as follows.

Conjecture 10.4. *For any $b \geq 1$, there exists a finite set Ω of semisimple subgroups of G defined over $\mathbb{Q}$ such that, if Z is a special subvariety of S, and $\deg(Z) \leq b$, then*

$$H^{\mathrm{der}} = \gamma F \gamma^{-1},$$

for some $\gamma \in \Gamma$ and some $F \in \Omega$.

Proof that Conjecture 10.4 implies Conjecture 10.3. By Conjecture 10.4, there exists a finite set Ω of semisimple subgroups of G defined over $\mathbb{Q}$, independent of Z, such that

$$H^{\mathrm{der}} = \gamma F \gamma^{-1},$$

for some $\gamma \in \Gamma$ and some $F \in \Omega$.

Let $P \in Z$ be a special point such that $\Delta(P)$ is minimal among all special points in Z and let $x \in X$ be a point lying above P such that $\mathrm{MT}(x)$ is contained in H. Therefore, Z is equal to the image of $F(\mathbb{R})^+ \cdot \gamma^{-1}x$ in $\Gamma\backslash X$. Furtherore, $\mathrm{MT}(\gamma^{-1}x)$ is contained in

$$G_F := F \cdot Z_G(F)^\circ$$

and, by [32], Lemme 3.3, if we denote by $\mathfrak{X}'$ the $G_F(\mathbb{R})$ conjugacy class of $\gamma^{-1}x$, we obtain a Shimura subdatum $(G_F, \mathfrak{X}')$ of $(G, \mathfrak{X})$.

Therefore, let X' denote the connected component $G_F(\mathbb{R})^+\gamma^{-1}x$ of $\mathfrak{X}'$ and let Γ' denote $\Gamma \cap G_F(\mathbb{Q})_+$, where $G_F(\mathbb{Q})_+$ denotes the subgroup of $G_F(\mathbb{Q})$ acting on X'. By [36], Proposition 3.21 and its proof, there exist only finitely many Γ' orbits of pre-special points in X' whose image in $\Gamma'\backslash X'$ has complexity at most b. Therefore, there exists $\lambda \in \Gamma'$ such that $\gamma^{-1}x = \lambda y$, where $y \in X'$ belongs to a finite set. We conclude that Z is equal to the image of

$$\Gamma F(\mathbb{R})^+\gamma^{-1}x = \Gamma F(\mathbb{R})^+\lambda y = \Gamma \lambda F(\mathbb{R})^+ y = \Gamma F(\mathbb{R})^+ y$$

in $\Gamma\backslash X$, which concludes the proof. □

11 Galois orbits

In [18], Habegger and Pila formulated a conjecture about Galois orbits of optimal points in $\mathbb{C}^n$ that in [17] they had been able to prove for so-called asymmetric curves. In [25], Orr generalized the result to asymmetric curves in $\mathcal{A}_g^2$.

Recall that $\mathrm{Sh}_K(G, \mathfrak{X})$ possesses a canonical model, defined over a number field E, which depends only on $(G, \mathfrak{X})$. Furthermore, S is defined over a finite abelian extension F of E. In particular, for any extension L of F contained in $\mathbb{C}$, it makes sense to say that a subvariety V of S is defined over L. Moreover, if V is such a subvariety, then $\mathrm{Gal}(\mathbb{C}/L)$ acts on the points of V.

If Z is a special subvariety of S and $\sigma \in \mathrm{Gal}(\mathbb{C}/F)$, then $\sigma(Z)$ is also a special subvariety of S and its complexity is $\Delta(Z)$. In particular, if V is a subvariety of S, as above, then $\mathrm{Gal}(\mathbb{C}/L)$ acts on $\mathrm{Opt}(V)$.

Conjecture 11.1 (large Galois orbits)**.** *Let V be a subvariety of S, defined over a finitely generated extension L of F contained in $\mathbb{C}$. There exist positive constants c_G and δ_G such that the following holds.*

If $P \in \mathrm{Opt}_0(V)$, then

$$\#\mathrm{Gal}(\mathbb{C}/L) \cdot P \geq c_G \Delta(\langle P \rangle)^{\delta_G}.$$

Remark 11.2. In the context of the André-Oort conjecture, there is the pioneering hypothesis that Galois orbits of special points should be large. See [12], Problem 14 for the formulation for special points in $\mathcal{A}_g$ and see [39], Theorem 2.1 for special points in a general Shimura variety. This hypothesis, which was verified by Tsimerman for special points of $\mathcal{A}_g$ [30] via progress on the Colmez conjecture due to Andreatta, Goren, Howard, Madapusi Pera, Yuan, and Zhang [1, 40], is now the only obstacle in an otherwise unconditional proof of the André-Oort conjecture. The conjecture is that there exist positive constants c and δ such that, for any special point $P \in S$,

$$\#\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \cdot P \geq c\Delta(P)^{\delta}.$$

Of course, this conjecture does not follow from Conjecture 11.1 because special points lying in V need not be optimal in V. However, the proof of the André-Oort conjecture only requires the bound for special points that are not contained in the positive dimensional special subvarieties contained in V i.e. special points contained in $\mathrm{Opt}_0(V)$ (see [7] for more details). Furthermore, since special points are defined over number fields, we may also assume in that case that V is defined over a finite extension of F. It follows that Conjecture 11.1 is sufficient to prove the André-Oort conjecture.

To prove Conjecture 8.4, however, one only requires the following hypothesis.

Conjecture 11.3. *Let V be a subvariety of S, defined over a finitely generated extension L of F contained in $\mathbb{C}$. There exist positive constants c_G and δ_G such that the following holds.*

If $P \in V^{\mathrm{oa}} \cap S^{[1+\dim V]}$, then

$$\#\mathrm{Gal}(\mathbb{C}/L) \cdot P \geq c_G\Delta(\langle P\rangle)^{\delta_G}.$$

Remark 11.4. Note that, if $P \in V^{\mathrm{oa}} \cap S^{[1+\dim V]}$, then $P \in \mathrm{Opt}_0(V)$. To see this, let W be a subvariety of V containing P such that $\delta(W) \leq \delta(P)$ i.e.

$$\dim\langle W\rangle - \dim W \leq \dim\langle P\rangle \leq \dim S - 1 - \dim V.$$

Therefore,

$$\dim W \geq 1 + \dim V + \dim\langle W\rangle - \dim S \geq 1 + \dim V + \dim\langle W\rangle_{\mathrm{ws}} - \dim S,$$

and so $\dim W = 0$, as $P \notin V^{\mathrm{an}}$, which implies that $W = P$, proving the claim. Therefore, Conjecture 11.3 follows from Conjecture 11.1, but the former would seem genuinely more tractable. Indeed, when S is an abelian variety and V is a subvariety defined over $\bar{\mathbb{Q}}$, Habegger [16] showed that the Néron-Tate height is bounded on $\bar{\mathbb{Q}}$-points of $V^{\mathrm{oa}} \cap S^{[\dim V]}$.

12 Further arithmetic hypotheses

The principal obstruction to applying the Pila-Wilkie counting theorems to our point counting problems (except for the availabliity of lower bounds for Galois orbits) is the ability to parametrize pre-special subvarieties of S using points of bounded height.

Definition 12.1. We say that a semisimple algebraic group defined over $\mathbb{Q}$ is of non-compact type if its almost-simple factors all have the property that their underlying real Lie group is not compact.

Let Ω be a (finite) set of representatives for the semisimple subgroups of G defined over $\mathbb{Q}$ of non-compact type modulo the equivalence relation

$$H_1 \sim H_2 \iff H_{2,\mathbb{R}} = gH_{1,\mathbb{R}}g^{-1}, \text{ for some } g \in G(\mathbb{R}).$$

Add the trivial group to Ω. Our needs can be encapsulated in the following conjecture. Recall that X is realised as a bounded symmetric domain in $\mathbb{C}^N$, for some $N \in \mathbb{N}$, which we identify with $\mathbb{R}^{2N}$. Henceforth, we fix an embedding of G into GL_n such that Γ is contained in $\mathrm{GL}_n(\mathbb{Z})$. We consider $\mathrm{GL}_n(\mathbb{R})$ as a subset of $\mathbb{R}^{n^2}$ in the natural way.

Conjecture 12.2 (cf. [18], Proposition 6.7)**.** *There exist positive constants d, $c_{\mathcal{F}}$, and $\delta_{\mathcal{F}}$ such that, if $z \in \mathcal{F}$, then the smallest pre-special subvariety of X containing z can be written $gF(\mathbb{R})^+g^{-1}x$, where $F \in \Omega$, and $g \in G(\mathbb{R})$ and $x \in X$ satisfy*

$$\mathrm{H}_d(g, x) \le c_{\mathcal{F}}\Delta(\langle\pi(z)\rangle)^{\delta_{\mathcal{F}}}.$$

This is seemingly the most natural generalization of the following theorem due to Orr and the first author on the heights of pre-special points, which plays a crucial role in the proof of the André-Oort conjecture.

Theorem 12.3 (cf. [8], Theorem 1.4)**.** *There exist positive constants d, $c_{\mathcal{F}}$ and $\delta_{\mathcal{F}}$ such that, if $z \in \mathcal{F}$ is a pre-special, then*

$$\mathrm{H}_d(z) \le c_{\mathcal{F}}\Delta(\pi(z))^{\delta_{\mathcal{F}}}.$$

We remark that the problem of finding d as in Conjecture 12.2 poses no obstacle in itself. Indeed a proof of the following theorem will appear in a forthcoming article of Borovoi and the authors.

Theorem 12.4. *There exists a positive constant d such that, for any two semisimple subgroups H_1 and H_2 of G defined over $\mathbb{Q}$ that are conjugate by an element of $G(\mathbb{R})$, there exists a number field K contained in $\mathbb{R}$ of degree at most d, and an element $g \in G(K)$, such that*

$$H_{2,K} = gH_{1,K}g^{-1}.$$

A nice feature of Conjecture 12.2 is that it implies Conjecture 10.3 that there are only finitely many special subvarieties of bounded complexity.

Lemma 12.5. *Conjecture 12.2 implies Conjecture 10.3.*

Proof. Let Z be a special subvariety of S such that $\Delta(Z) \le b$ and let $P \in Z$ be such that $\langle P\rangle = Z$. Let $z \in \mathcal{F}$ be such that $\pi(z) = P$ and let X_H be the smallest pre-special subvariety of X containing z. Then $\pi(X_H) = Z$ and, by Conjecture 12.2, $X_H = gFg^{-1}x$, where $F \in \Omega$, and $g \in G(\mathbb{R})$ and $x \in X$ satisfy

$$\mathrm{H}_d(g, x) \le c_{\mathcal{F}}\Delta(Z)^{\delta_{\mathcal{F}}} \le c_{\mathcal{F}}b^{\delta_{\mathcal{F}}}.$$

The claim follows, therefore, from the fact that there are only finitely many algebraic numbers of bounded degree and height.

□

Another, albeit longer, approach to our point counting problems can be given by replacing Conjecture 12.2 with two related conjectures, although we will have to additionally assume Conjecture 10.3 in this case. We will also rely on the fact that Theorem 9.1 is uniform in families. The advantage is that the following two conjectures are seemingly more accessible.

Conjecture 12.6. *For any $\kappa > 0$, there exists a positive constant c_κ such that, if Z is a special subvariety of S, then there exists a semisimple subgroup H of G defined over $\mathbb{Q}$ of non-compact type, and an extension L of F satisfying*

$$[L : F] \leq c_\kappa \Delta(Z)^\kappa,$$

such that, for any $\sigma \in \mathrm{Gal}(\mathbb{C}/L)$,

$$\sigma(Z) = \pi(H(\mathbb{R})^+ x_\sigma),$$

where $H(\mathbb{R})^+ x_\sigma$ is a pre-special subvariety of X intersecting $\mathcal{F}$.

Recall that, for an abelian variety A, defined over a field K, every abelian subvariety of A can be defined over a fixed, finite extension of K. The analogue of Conjecture 12.6 is, therefore, trivial. In a Shimura variety, one hopes that the degrees of fields of definition of strongly special subvarieties grow as in Conjecture 12.6. If this were true, Conjecture 12.6 for strongly special subvarieties would follow easily.

Our final conjecture is also inspired by the abelian setting.

Conjecture 12.7 (cf. [18], Lemma 3.2)**.** *There exist positive constants c_Γ and δ_Γ such that, if X_H is a pre-special subvariety of X intersecting $\mathcal{F}$ and $z \in \mathcal{F}$ belongs to ΓX_H, then $z \in \gamma X_H$, where $\gamma \in \Gamma$ satisfies*

$$\mathrm{H}_1(\gamma) \leq c_\Gamma \deg(\pi(X_H))^{\delta_\Gamma}.$$

Conjecture 12.7 has the following useful consequence.

Lemma 12.8. *Assume that Conjecture 12.7 holds.*

There exist positive constants d, c_{H}, and δ_{H} such that, if $H(\mathbb{R})^+ x$ is a pre-special subvariety of X intersecting $\mathcal{F}$, then

$$H(\mathbb{R})^+ x = H(\mathbb{R})^+ y$$

where $\mathrm{H}_d(y) \leq c_{\mathrm{H}} \Delta(\pi(H(\mathbb{R})^+ x))^{\delta_{\mathrm{H}}}$.

Proof. Let d, $c_{\mathcal{F}}$, and $\delta_{\mathcal{F}}$ be the positive constants afforded to us by Theorem 12.3, and let c_Γ and δ_Γ be the positive constants afforded to us by Conjecture 12.7.

Let $x' \in \Gamma H(\mathbb{R})^+ x \cap \mathcal{F}$ denote a pre-special point such that $\pi(x')$ is of minimal complexity among the special points of $\pi(H(\mathbb{R})^+ x)$. By Theorem 12.3, we have

$$\mathrm{H}_d(x') \leq c_{\mathcal{F}} \Delta(\pi(H(\mathbb{R})^+ x)^{\delta_{\mathcal{F}}}.$$

On the other hand, by Conjecture 12.7, $x' \in \gamma H(\mathbb{R})^+ x$, where

$$\mathrm{H}_1(\gamma) \leq c_\Gamma \Delta(\pi(H(\mathbb{R})^+ x)^{\delta_\Gamma}.$$

It follows easily from the properties of heights that there exist positive constants c and δ depending only on the fixed data such that

$$\mathrm{H}_d(\gamma^{-1} x') \leq c \mathrm{H}_1(\gamma)^\delta \mathrm{H}_d(x')^\delta.$$

Therefore, the previous remarks show that

$$y := \gamma^{-1} x' \in H(\mathbb{R})^+ x$$

satisfies the requirements of the lemma. □

We will now verify the arithmetic conjectures stated above in an arbitrary product of modular curves.

13 Products of modular curves

Our definition of a Shimura variety allows for the possibility that S might be a product of modular curves. In that case $G = \mathrm{GL}_2^n$, where n is the number of modular curves, and $\mathfrak{X}$ is the $G(\mathbb{R})$ conjugacy class of the morphism $\mathbb{S} \to G_{\mathbb{R}}$ given by

$$a + ib \mapsto \left(\begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \ldots, \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \right).$$

We let X denote the $G(\mathbb{R})^+$ conjugacy class of this morphism, which one identifies with the n-th cartesian power $\mathbb{H}^n$ of the upper half-plane $\mathbb{H}$.

For our purposes, we can and do suppose that Γ is equal to $\mathrm{SL}_2(\mathbb{Z})^n$ and we let $\mathcal{F}$ denote a fundamental set in X for the action of Γ, equal to the n-th cartesian power of a fundamental set $\mathcal{F}_{\mathbb{H}}$ in $\mathbb{H}$ for the action of $\mathrm{SL}_2(\mathbb{Z})$. Note that, as explained in [24] Section 1.3, we can and do choose $\mathcal{F}_{\mathbb{H}}$ in the image of a Siegel set. Via the j-function applied to each factor of $\mathbb{H}^n$, the quotient $\Gamma\backslash X$ is isomorphic to the algberaic variety $\mathbb{C}^n$. Special subvarieties have the following well-documented description.

Proposition 13.1 (cf. [10], Proposition 2.1)**.** *Let $I = \{1, \ldots, n\}$. A subvariety Z of $\mathbb{C}^n$ is a special subvariety if and only if there exists a partition $\Omega = (I_1, \ldots, I_t)$ of I, with $|I_i| = n_i$, such that Z is equal to the product of subvarieties Z_i of $\mathbb{C}^{n_i}$, where, either*

- ⋆ *I_i is a one element set and Z_i is a special point, or*
- ⋆ *Z_i is the image of $\mathbb{H}$ in $\mathbb{C}^{n_i}$ under the map sending $\tau \in \mathbb{H}$ to the image of $(g_j \cdot \tau)_{j \in I_i}$ in $\mathbb{C}^{n_i}$ for elements $g_j \in \mathrm{GL}_2(\mathbb{Q})^+$.*

First note that Conjecture 12.2 for $\mathbb{C}^n$ follows from Proposition 6.7 of [18]. Hence, we will now verify Conjecture 12.6 and Conjecture 12.7 in that setting.

Proof of Conjecture 12.6 for $\mathbb{C}^n$. Let Z be a special subvariety of $\mathbb{C}^n$, equal to a product of special subvarieties Z_i of $\mathbb{C}^{n_i}$, as above. Without loss of generality, we may assume that the product contains only one factor and, by Theorem 12.3, we may assume that it is not a special point. Therefore, Z is equal to the image of $\mathbb{H}$ in $\mathbb{C}^n$ under the map sending $\tau \in \mathbb{H}$ to the image of $(g_j \cdot \tau)_{j=1}^n$ in $\mathbb{C}^n$ for elements $g_j \in \mathrm{GL}_2(\mathbb{Q})^+$.

In other words, we have a morphism of Shimura data from $(\mathrm{GL}_2, \mathbb{H}^{\pm})$ to $(G, \mathfrak{X})$, where $\mathbb{H}^{\pm}$ is the union of the upper and lower half-planes (or, rather, the conjugacy class we associate with it, as above), induced by the morphism

$$\mathrm{GL}_2 \to \mathrm{GL}_2^n : g \mapsto (g_j g g_j^{-1})_{j=1}^n,$$

such that Z is equal to the image of $\mathbb{H} \times \{1\}$ under the corresponding morphism

$$\mathrm{Sh}_K(\mathrm{GL}_2, \mathbb{H}^{\pm}) \to \mathrm{Sh}_{\mathrm{GL}_2(\hat{\mathbb{Z}})^n}(G, \mathfrak{X}), \tag{13.1.1}$$

where K is the product of the groups

$$K_p := g_1 \mathrm{GL}_2(\mathbb{Z}_p) g_1^{-1} \cap \cdots \cap g_n \mathrm{GL}_2(\mathbb{Z}_p) g_n^{-1}$$

over all primes p.

Since (13.1.1) is defined over $E(\mathrm{GL}_2, \mathbb{H}^{\pm}) = \mathbb{Q}$, it suffices to bound the size of

$$\pi_0(\mathrm{Sh}_K(\mathrm{GL}_2, \mathbb{H}^{\pm})),$$

which, by [22], Theorem 5.17, is in bijection with

$$\mathbb{Q}_{>0}\backslash\mathbb{A}_f^\times/\nu(K),$$

where ν is the determinant map on GL_2. However, since $\mathbb{A}_f^\times$ is equal to the direct product $\mathbb{Q}_{>0}\hat{\mathbb{Z}}^\times$, it suffices to bound the size of $\hat{\mathbb{Z}}^\times/\nu(K)$.

To that end, let Σ denote the (finite) set of primes p such that $g_j\notin\mathrm{GL}_2(\mathbb{Z}_p)$, for some $j\in\{1,\dots,n\}$. In particular,

$$[\hat{\mathbb{Z}}^\times:\nu(K)]=\prod_{p\in\Sigma}[\mathbb{Z}_p^\times:\nu(K_p)]$$

and, since K_p contains the elements $\mathrm{diag}(a,a)$, where $a\in\mathbb{Z}_p^\times$,

$$[\mathbb{Z}_p^\times:\nu(K_p)]\leq[\mathbb{Z}_p^\times:\mathbb{Z}_p^{\times 2}]\leq 4,$$

where $\mathbb{Z}_p^{\times 2}$ denotes the squares in $\mathbb{Z}_p^\times$.

On the other hand, by [6],

$$\deg(Z)\geq\prod_{p\in\Sigma}p,$$

and the conjecture follows easily from the following classical fact regarding primorials.

Lemma 13.2. *Let $n\in\mathbb{N}$. The product of the first n prime numbers is equal to*

$$e^{(1+o(1))n\log n}.$$

□

Proof of Conjecture 12.7 for $\mathbb{C}^n$. Let X_H be a product of spaces X_i equal to either a pre-special point or to the image of $\mathbb{H}$ in $\mathbb{H}^{n_i}$ given by the map sending τ to $(g_j\cdot\tau)_{j=1}^{n_i}$ for elements $g_j\in\mathrm{GL}_2(\mathbb{Q})^+$. Without loss of generality, we may assume that the product contains only one factor $X_1=X$.

If X is a pre-special point contained in $\mathcal{F}$, then the claim follows from the fact that

$$\{\gamma\in\Gamma:\gamma\mathcal{F}\cap\mathcal{F}\neq\emptyset\}$$

is finite. Therefore, assume that X is equal to the image of $\mathbb{H}$ in $\mathbb{H}^n$ given by the map sending τ to $(g_j\cdot\tau)_{j=1}^n$ for elements $g_j\in\mathrm{GL}_2(\mathbb{Q})^+$. We can and do assume that g_1 is equal to the identity element and that all of the g_j have coprime integer entries.

As in the statement of Conjecture 12.7, we assume that X intersects $\mathcal{F}$, and we let $x\in\mathcal{F}\cap X$. Therefore,

$$x=(g_j\tau_x)_{j=1}^n,$$

where $\tau_x\in\mathcal{F}_{\mathbb{H}}$ and, by [24], Theorem 1.2 (cf. [17], Lemma 5.2), $\mathrm{H}_1(g_j)\leq c_1\det(g_j)^2$, for all $j\in\{1,\dots,n\}$, where c_1 is a positive constant not depending on Z. In particular,

$$\mathrm{H}_1(g_j^{-1})\leq\det(g_j)\cdot\mathrm{H}_1(g_j)\leq c_1\det(g_j)^3,$$

for each $j\in\{1,\dots,n\}$.

Now, let $z:=(z_j)_{j=1}^n\in\mathcal{F}$ be a point belonging to ΓX. For each $j\in\{1,\dots,n\}$,

$$z_j=\gamma_j g_j g\tau_x,$$

for some $g \in \mathrm{GL}_2(\mathbb{R})^+$ and some $\gamma_j \in \mathrm{SL}_2(\mathbb{Z})$. Therefore, let

$$\Lambda := \bigcap_{j=1}^{n} g_j^{-1}\mathrm{SL}_2(\mathbb{Z})g_j$$

and let $\mathcal{C}$ denote a set of representatives in $\mathrm{SL}_2(\mathbb{Z})$ for $\Lambda\backslash\mathrm{SL}_2(\mathbb{Z})$. Note that, if we define

$$m_j := \det(g_j),$$

then, for any multiple N of m_j, the principal congruence subgroup $\Gamma(N)$ is contained in $g_j^{-1}\mathrm{SL}_2(\mathbb{Z})g_j$. In particular, if we define N to be the product of the m_j, then $\Gamma(N)$ is contained in Λ. It follows that any subset of $\mathrm{SL}_2(\mathbb{Z})$ mapping bijectively to $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ contains a set $\mathcal{C}$, as above. Via the procedure outlined in [9], Exercise 1.2.2, it is straightforward to verify that we can (and do) choose $\mathcal{C}$ such that, for any $c \in \mathcal{C}$,

$$H(c) \leq 7N^5$$

(though we certainly do not claim that this is the best possible bound).

The union

$$\bigcup_{c\in\mathcal{C}} c\mathcal{F}_{\mathbb{H}}$$

constitutes a fundamental set in $\mathbb{H}$ for the action of Λ. Hence, there exists $c \in \mathcal{C}$ and $\lambda_g \in \Lambda$ such that

$$c^{-1}\lambda_g g\tau_x \in \mathcal{F}_{\mathbb{H}}.$$

Furthermore, for each $j \in \{1,\dots,n\}$, we can write $\lambda_g = g_j^{-1}\lambda_j g_j$, for some $\lambda_j \in \mathrm{SL}_2(\mathbb{Z})$ and, hence,

$$z_j = \gamma_j g_j g\tau_x = \gamma_j g_j \lambda_g^{-1} c \cdot c^{-1}\lambda_g g\tau_x = \gamma_j\lambda_j^{-1} g_j c \cdot c^{-1}\lambda_g g\tau_x.$$

Therefore, by [24], Theorem 1.2, we have $\mathrm{H}_1(\gamma_j\lambda_j^{-1}g_jc) \leq c_1 m_j^2$, for all $j \in \{1,\dots,n\}$.

We write

$$\mathrm{H}_1(\gamma_j\lambda_j^{-1}) = \mathrm{H}_1(\gamma_j\lambda_j^{-1}g_jc\cdot c^{-1}g_j^{-1}) \leq c_2\mathrm{H}_1(\gamma_j\lambda_j^{-1}g_jc)^{\delta}\mathrm{H}_1(c^{-1})^{\delta}\mathrm{H}_1(g_j^{-1})^{\delta},$$

where c_2 and δ are positive constants not depending on Z, and we obtain

$$\mathrm{H}_1(\gamma_j\lambda_j^{-1}) \leq c_3(Nm_j)^{5\delta},$$

for each $j \in \{1,\dots,n\}$, where c_3 is a positive constant not depending on Z.

Conversely, by [6], §2,

$$\deg(Z) \geq [\mathrm{SL}_2(\mathbb{Z}) : \Lambda]$$

and, by writing the g_j in Smith normal form i.e.

$$g_j = \gamma_j^{(1)} \begin{pmatrix} 1 & 0 \\ 0 & m_j \end{pmatrix} \gamma_j^{(2)},$$

where $\gamma_j^{(1)}, \gamma_j^{(2)} \in \mathrm{SL}_2(\mathbb{Z})$, we conclude that

$$[\mathrm{SL}_2(\mathbb{Z}) : \Lambda] \geq [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(m_j)],$$

for all $j \in \{1,\dots,n\}$. Since

$$[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(m_j)] \geq m_j,$$

the result follows. □

14 Main results (part 2): Conditional solutions to the counting problems

We conclude by demonstrating how our arithmetic conjectures might be used to resolve the counting problems stated in Section 8. In our applications of the counting theorem, we will need the following.

Lemma 14.1. *Let $\beta : [0,1] \to G(\mathbb{R}) \times X$ be semi-algebraic. Then $\mathrm{Im}(\beta)$ is contained in a complex algebraic subset B of $G(\mathbb{C}) \times \mathbb{C}^N$ of dimension at most 1.*

Proof. Let Y denote the real Zariski closure of $\mathrm{Im}(\beta)$ in $G(\mathbb{R}) \times \mathbb{R}^{2N}$. In particular, $\dim Y \leq 1$. Without loss of generality, we can and do assume that Y is irreducible. If Y is a point then there is nothing to prove. Therefore, we can and do assume that Y is an irreducible real algebraic curve. In particular, the complexification $Y_{\mathbb{C}}$ of Y in $G(\mathbb{C}) \times \mathbb{C}^{2N}$ is an irreducible complex algebraic curve.

Let $g_1, \ldots, g_{n^2}, x_1, y_1, \ldots, x_N, y_N$ denote the real coordinate functions on $G(\mathbb{R}) \times X$ and let $z_j = x_j + iy_j$ denote the coordinate functions on $\mathbb{C}^N = \mathbb{R}^{2N}$. If all of the coordinates functions on $\mathbb{R}^{2N}$ are constant on Y, the result is obvious. Therefore, without loss of generality, we can and do assume that x_1 is not constant on Y.

We claim that each of the coordinate functions $x_2, y_2, \ldots, x_N, y_N$ on $\mathbb{C}^{2N}$ is algebraic over the field $\mathbb{C}(z_1)$, considered as a field of functions on $Y_{\mathbb{C}}$. To see this, note that z_1 is non-constant on $Y_{\mathbb{C}}$, and so $\mathbb{C}(z_1)$ has transcendence degree at least 1. On the other hand, $\mathbb{C}(z_1)$ is contained in $\mathbb{C}(x_1, y_1)$, which is algebraic over $\mathbb{C}(x_1)$.

In particular, each of the functions $x_2 + iy_2, \ldots, x_N + iy_N$ is algebraic over the field $\mathbb{C}(z_1)$. It follows that, for each $j \geq 2$, there exists a polynomial $f_j(z_1, z_j) \in \mathbb{C}[z_1, z_j]$, non-trivial in z_j, such that $f_j(z_1, z_j) = 0$ on Y. Similarly, for each $k = 1, \ldots, n^2$, there exists a polynomial $f_j(z_1, g_k) \in \mathbb{C}[z_1, g_k]$, non-trivial in g_k, such that $f_k(z_1, g_k) = 0$ on Y. In particular, Y is contained in the vanishing locus of the f_j and the f_k, which define a complex algebraic curve in $G(\mathbb{C}) \times \mathbb{C}^N$. □

We denote by $X^\vee$ the complex dual of X, which is a complex algebraic variety on which $G(\mathbb{C})$ acts via an algebraic morphism

$$G(\mathbb{C}) \times X^\vee \to X^\vee.$$

Furthermore, X naturally embeds into $X^\vee$ and the embedding factors through an embedding of $\mathbb{C}^N$ i.e. the Harish-Chandra realization, into $X^\vee$. We could have defined subvarieties of X using $X^\vee$ in the place of $\mathbb{C}^N$ but, in fact, the two notions coincide. If we have a decomposition $G^{\mathrm{ad}} = G_1 \times G_2$, and thus $X = X_1 \times X_2$, then we have a natural decomposition

$$X^\vee = X_1^\vee \times X_2^\vee.$$

Furtherore, if $(H, \mathfrak{X}_H)$ denotes a Shimura subdatum of $(G, \mathfrak{X})$ and X_H is a connected component of $\mathfrak{X}_H$ contained in X, then $X_H^\vee$ is naturally contained in $X^\vee$. We refer the reader to [35], Section 3 for more details.

Theorem 14.2. *Assume that Conjecture 11.1 holds and assume that either*

- ⋆ *Conjecture 12.2 holds or*
- ⋆ *Conjectures 10.3, 12.6, and 12.7 hold.*

Then Conjecture 8.2 is true for curves i.e. if V is a curve contained in S, then the set $\mathrm{Opt}_0(V)$ is finite.

Proof. We will assume that Conjecture 12.2 holds. The proof in the case that Conjectures 10.3, 12.6, and 12.7 hold is very similar, hence we omit it. To elucidate the use of Conjectures 10.3, 12.6, and 12.7 we will use them in the proof of Theorem 14.3, at the expense of making the proof longer. We suffer no loss of generality if we assume, as we will, that V is Hodge generic.

Let Ω denote a finite set of semisimple subgroups of G defined over $\mathbb{Q}$ as in Section 12 and let d, $c_{\mathcal{F}}$, and $\delta_{\mathcal{F}}$ be the constants afforded to us by Conjecture 12.2. Let L be a finitely generated extension of F contained in $\mathbb{C}$ over which V is defined and let c_G and δ_G be the constants afforded to us by Conjecture 11.1. Let $\kappa := 2\delta_G/3\delta_{\mathcal{F}}$.

We claim that there exists a positive constant c such that, for any $P \in \mathrm{Opt}_0(V)$, we have

$$\#\mathrm{Gal}(\mathbb{C}/L)\cdot P \leq c c_{\mathcal{F}}^{\kappa}\Delta(\langle P\rangle)^{\kappa\delta_{\mathcal{F}}}.$$

This would be sufficient to prove Theorem 14.2 since then, by Conjecture 11.1, we obtain

$$c_G\Delta(\langle P\rangle)^{\delta_G} \leq \#\mathrm{Gal}(\mathbb{C}/L)\cdot P \leq c c_{\mathcal{F}}^{\kappa}\Delta(\langle P\rangle)^{\kappa\delta_{\mathcal{F}}}$$

and, rearranging this expression, we obtain

$$\Delta(\langle P\rangle) \leq (c_3 c_{\mathcal{F}}^{\kappa} c_G^{-1})^3,$$

which is a bound independent of P. We remind the reader that P is one of only finitely many irreducible components of $\langle P\rangle \cap V$. Hence, Theorem 14.2 would follow from Lemma 12.5 and it remains only, therefore, to prove the claim.

To that end, for each $\sigma \in \mathrm{Gal}(\mathbb{C}/L)$, let $z_\sigma \in \mathcal{V}$ be a point in $\pi^{-1}(\sigma(P))$. Therefore, by Conjecture 12.2, the smallest pre-special subvariety of X containing z_σ can be written $g_\sigma F_\sigma(\mathbb{R})^+ g_\sigma^{-1} x_\sigma$, where $F_\sigma \in \Omega$, and $g_\sigma \in G(\mathbb{R})$ and $x_\sigma \in X$ satisfy

$$\mathrm{H}_d(g_\sigma, x_\sigma) \leq c_{\mathcal{F}}\Delta(\langle\sigma(P)\rangle)^{\delta_{\mathcal{F}}} = c_{\mathcal{F}}\Delta(\langle P\rangle)^{\delta_{\mathcal{F}}}.$$

Without loss of generality, we can and do assume that $F := F_\sigma$ is fixed. Therefore, for each $\sigma \in \mathrm{Gal}(\mathbb{C}/L)$, the tuple $(g_\sigma, x_\sigma, z_\sigma)$ belongs to the definable set D of tuples

$$(g, x, z) \in G(\mathbb{R}) \times X \times X \subseteq \mathbb{R}^{n^2+2N} \times \mathbb{R}^{2N},$$

such that $z \in \mathcal{V} \cap gF(\mathbb{R})^+ g^{-1}x$ and $x(\mathbb{S}) \subseteq gG_F g^{-1}$. We consider D as a family over a point in an omitted parameter space and choose for c the constant $c(D, d, \kappa)$ afforded to us by Theorem 9.1 applied to D. Since Ω is finite, we can and do assume that c does not depend on F. We let Σ denote the union over $\mathrm{Gal}(\mathbb{C}/L)$ of the tuples $(g_\sigma, x_\sigma, z_\sigma) \in D$. In particular, Σ is contained in the subset

$$D(d, c_{\mathcal{F}}\Delta(\langle P\rangle)^{\delta_{\mathcal{F}}}).$$

Let π_1 and π_2 be the projection maps from $\mathbb{R}^{n^2+2N} \times \mathbb{R}^{2N}$ to $\mathbb{R}^{n^2+2N}$ and $\mathbb{R}^{2N}$, respectively, and suppose, for the sake of obtaining a contradiction, that

$$\#\mathrm{Gal}(\mathbb{C}/L)\cdot P = \#\pi_2(\Sigma) > c c_{\mathcal{F}}^{\kappa}\Delta(\langle P\rangle)^{\kappa\delta_{\mathcal{F}}}.$$

Then, by Theorem 9.1, there exists a continuous definable function

$$\beta : [0,1] \to D,$$

such that $\beta_1 := \pi_1 \circ \beta$ is semi-algebraic, $\beta_2 := \pi_2 \circ \beta$ is non-constant, and $\beta(0) \in \Sigma$. Let $z_0 := \beta_2(0)$ and let $P_0 := \pi(z_0)$. To obtain a contradiction, we will closely imitate arguments found in [25].

Since β_2 is continuous, it follows from the Global Decomposition Theorem (see [15], p172) that there exists $0 < t \leq 1$ and an irreducible analytic component V_1 of $\pi^{-1}(V)$ that contains $\beta_2([0,t])$. By [34], Theorem 1.3 (the inverse Ax-Lindemann conjecture), $\langle V_1\rangle_{\mathrm{Zar}}$ is pre-weakly special and so, since V is Hodge generic in S, we can decompose $G^{\mathrm{ad}} = G_1 \times G_2$, and thus $X = X_1 \times X_2$, so that

$$\langle V_1\rangle_{\mathrm{Zar}} = X_1 \times \{x_2\},$$

where $x_2 \in X_2$ is Hodge generic. By abuse of notation, we denote by π_2 both the projection from G to G_2 and from $X^\vee$ to $X_2^\vee$.

Note that, for any $(g,x) \in \mathrm{Im}(\beta_1)$, we have $(g^{-1}x)(\mathbb{S}) \subseteq G_{F,\mathbb{R}}$. If we write G'_F for the largest normal subgroup of G_F of non-compact type, then the properties of Shimura data imply that $g^{-1}x$ factors through $G'_{F,\mathbb{R}}$ and, if we write $\mathfrak{X}'$ for the $G'_F(\mathbb{R})$ conjugacy class of $g^{-1}x$ in $\mathfrak{X}$, then, by [32], Lemme 3.3, $(G'_F,\mathfrak{X}')$ is a Shimura subdatum of $(G,\mathfrak{X})$. Furthermore, by [36], Lemma 3.7, the number of Shimura subdata $(G'_F,\mathfrak{Y})$ of $(G,\mathfrak{X})$ is finite and, by [22], Corollary 5.3, the number of connected components Y of $\mathfrak{Y}$ is also finite. It follows that, after possibly replacing t, we can and do assume that $g^{-1}x$ belongs to one such component Y, which we write as $Y_1 \times Y_2$, such that $F(\mathbb{R})^+$ acts transitively on Y_1. We let p_2 denote the projection from $Y^\vee$ to $Y_2^\vee$.

Let B denote the complex algebraic subset of $G(\mathbb{C}) \times X^\vee$ of dimension at most 1 containing $\mathrm{Im}(\beta_1)$ afforded to us by Lemma 14.1. For any $(g,x) \in B$, we have $g^{-1}x \in Y^\vee$.

Let $\overline{V}_1$ denote the Zariski closure of V_1 in $X^\vee$ and consider the complex algebraic set

$$W_B := \{(g,x,y) \in B \times Y^\vee : p_2(y) = p_2(g^{-1}x),\ gy \in \overline{V}_1\}.$$

Let V_B denote the Zariski closure in $X^\vee$ of the set

$$\{gy : (g,x,y) \in W_B\}.$$

Since the latter is the image of W_B under an algebraic morphism, we have $\dim V_B \leq \dim W_B$.

Since V_1 is an irreducible complex analytic curve having uncountable intersection with V_B, it follows that V_1 is contained in V_B. Therefore, $\langle V_1\rangle_{\mathrm{Zar}}$ is contained in V_B also, and so

$$\dim X_1 = \dim\langle V_1\rangle_{\mathrm{Zar}} \leq \dim V_B \leq W_B. \tag{14.2.1}$$

Now, for each $(g,x) \in B$, consider the fibre $W_{(g,x)}$ of W_B over (g,x) i.e. the set

$$\{y \in Y^\vee : p_2(y) = p_2(g^{-1}x),\ \pi_2(y) = \pi_2(g)^{-1}x_2\}.$$

Since $P_0 \in V$, it follows that $\pi_2(F) = G_2$ and so, for any $y \in Y_2^\vee$, the natural projection

$$Y_1^\vee \times \{y\} \to X_2^\vee$$

is an equivariant morphism of $F(\mathbb{C})$-homogeneous spaces. In particular, its fibres are equidimensional of dimension

$$\dim Y_1^\vee - \dim X_2^\vee = \dim Y_1 - \dim X_2.$$

Since $W_{(g,x)}$ is contained in such a fibre, we have

$$\dim W_{(g,x)} \leq \dim Y_1 - \dim X_2 \leq \dim X - 2 - \dim X_2 = \dim X_1 - 2,$$

where we use the fact that $P_0 \in \mathrm{Opt}_0(V)$, hence,

$$\dim Y_1 = \delta(P_0) \leq \delta(V) - 1 = \dim X - 2.$$

Since this holds for all $(g,x) \in B$ and $\dim B \leq 1$, we conclude that

$$\dim W_B \leq \dim X_1 - 1,$$

which contradicts (14.2.1). $\square$

Of course, Theorem 14.2 is not really satisfactory. One would hope that, for V of arbitrary dimension, a path such as β would yield, via the weak hyperbolic Ax-Schanuel conjecture, a positive dimensional subvariety of V, containing a conjugate of P, having defect at most $\delta(P)$, thus contradicting the optimality of P. However, the authors haven't been able to carry out this procedure. Instead, the very same idea appears to work when one attempts to contradict the membership of a point in the open-anomalous locus. The difference is that we are only required to bound the weakly special defect, as opposed to the defect itself.

Theorem 14.3. *Assume that Conjecture 11.3 holds and assume that the weak hyperbolic Ax-Schanuel conjecture is true. Assume also that, either*

- ⋆ *Conjecture 12.2 holds, or*
- ⋆ *Conjectures 10.3, 12.6, and 12.7 hold.*

Then, Conjecture 8.4 is true i.e. if V is a subvariety of S, then the set

$$V^{\mathrm{oa}} \cap S^{[1+\dim V]}$$

is finite.

Proof. We will assume that Conjectures 10.3, 12.6, and 12.7 hold. The proof in the case that Conjecture 12.2 holds is very similar, hence we omit it. We used Conjecture 12.2 in the proof of Theorem 14.2.

Let Ω denote a finite set of semisimple subgroups of G defined over $\mathbb{Q}$ as in Section 12. Let c_Γ and δ_Γ be the constants afforded to us by Conjecture 12.7, let $\kappa := \delta_G/3$, and let c_κ be the constant afforded to us by Conjecture 12.6. Let d, c_{H}, and δ_{H} be the constants afforded to us by Lemma 12.8, and let

$$c := \max\{c_{\mathrm{H}}, c_\Gamma\} \text{ and } \delta := \max\{\delta_{\mathrm{H}}, \delta_\Gamma\}.$$

Let L' be a finitely generated extension of F contained in $\mathbb{C}$ over which V is defined and let c_G and δ_G be the constants afforded to us by Conjecture 11.1. Let

$$P \in V^{\mathrm{oa}} \cap S^{[1+\dim V]}$$

and let L and H be, respectively, the field extension of F and the semisimple subgroup of G defined over $\mathbb{Q}$ of non-compact type afforded to us by Conjecture 12.6 applied to $\langle P\rangle$. Replace L by its compositum with L'. In particular, we have

$$[L : L'] \leq c_\kappa \Delta(\langle P\rangle)^\kappa.$$

We claim that there exists a positive constant c_3, independent of P, such that

$$\#\mathrm{Gal}(\mathbb{C}/L) \cdot P \leq c_3 c^{\frac{\kappa}{\delta}} \Delta(\langle P\rangle)^\kappa.$$

This would be sufficent to prove Theorem 14.3 since then, by Conjecture 11.3, we obtain

$$\frac{c_G}{c_\kappa}\Delta(\langle P\rangle)^{2\kappa} \leq \frac{1}{[L : L']}\#\mathrm{Gal}(\mathbb{C}/L') \cdot P = \#\mathrm{Gal}(\mathbb{C}/L) \cdot P \leq c_3 c^{\frac{\kappa}{\delta}} \Delta(\langle P\rangle)^\kappa.$$

and, rearranging this expression, we obtain

$$\Delta(\langle P\rangle) \leq (c_3 c^{\frac{\kappa}{\delta}} c_\kappa c_G^{-1})^{\frac{1}{\kappa}},$$

which is a bound independent of P. We remind the reader that, as explained in Remark 11.4, $P \in \mathrm{Opt}_0(V)$ and, therefore, P is one of only finitely many irreducible components of $\langle P\rangle \cap V$. Hence, Theorem 14.3 would follow from Conjecture 10.3 and it remains only, therefore, to prove the claim.

By Conjecture 12.6, for each $\sigma \in \mathrm{Gal}(\mathbb{C}/L)$,

$$\sigma(\langle P\rangle) = \pi(H(\mathbb{R})^+ x_\sigma),$$

where $H(\mathbb{R})^+ x_\sigma$ is a pre-special subvariety of X intersecting $\mathcal{F}$. By Lemma 12.8, we can and do assume that

$$\mathrm{H}_d(x_\sigma) \leq c_{\mathrm{H}} \Delta(\sigma(\langle P\rangle))^{\delta_{\mathrm{H}}} = c_{\mathrm{H}} \Delta(\langle P\rangle)^{\delta_{\mathrm{H}}}.$$

We let $z_\sigma \in \mathcal{V}$ be a point in $\pi^{-1}(\sigma(P))$, so that

$$z_\sigma \in \Gamma H(\mathbb{R})^+ x_\sigma$$

and so, by Corollary 12.7, there exists $\gamma_\sigma \in \Gamma$ satisfying

$$\mathrm{H}_1(\gamma_\sigma) \leq c_\Gamma \Delta(\langle P\rangle)^{\delta_\Gamma}$$

such that $z_\sigma \in \gamma_\sigma H(\mathbb{R})^+ x_\sigma$.

By definition, there exists $F \in \Omega$ and $g \in G(\mathbb{R})$ such that $H_\mathbb{R}$ is equal to $gF_\mathbb{R}g^{-1}$. In particular, for each $\sigma \in \mathrm{Gal}(\mathbb{C}/L)$, the tuple $(g, (\gamma_\sigma, x_\sigma), z_\sigma)$ belongs to the definable family D of tuples

$$(g, (\gamma, x), z) \in G(\mathbb{R}) \times [G(\mathbb{R}) \times X] \times X \subseteq \mathbb{R}^{n^2} \times \mathbb{R}^{n^2+2N} \times \mathbb{R}^{2N},$$

parametrised by $G(\mathbb{R})$, such that

$$z \in \mathcal{V} \cap \gamma g F(\mathbb{R})^+ g^{-1} x, \text{ and } x(\mathbb{S}) \subseteq gG_F g^{-1}.$$

We choose, then, for c_3 the constant $c(D, d, \kappa/\delta)$ afforded to us by Theorem 9.1 applied to D. Since, Ω is finite, we can and do assume that c_3 does not depend on F. We let Σ denote the union over $\mathrm{Gal}(\mathbb{C}/L)$ of the tuples $((\gamma_\sigma, x_\sigma), z_\sigma) \in D_g$ (to use the notation of Section 9). In particular, Σ is contained in the subset

$$D_g(d, c\Delta(Z)^\delta).$$

Let π_1 and π_2 be the projection maps from $\mathbb{R}^{n^2+2N} \times \mathbb{R}^{2N}$ to $\mathbb{R}^{n^2+2N}$ and $\mathbb{R}^{2N}$, respectively, and suppose, for the sake of obtaining a contradiction, that

$$\#\mathrm{Gal}(\mathbb{C}/L) \cdot P = \#\pi_2(\Sigma) > c_3 c^{\frac{\kappa}{\delta}} \Delta(Z)^\kappa.$$

Then, by Theorem 9.1, there exists a continuous definable function

$$\beta : [0, 1] \to D_g,$$

such that $\beta_1 := \pi_1 \circ \beta$ is semi-algebraic, $\beta_2 := \pi_2 \circ \beta$ is non-constant, and $\beta(0) \in \Sigma$. Let $z_0 := \beta_2(0)$ and $(\gamma_0, x_0) := \beta_1(0)$. Denote by P_0 the point $\pi(z_0)$ and denote by X_0 the special subvariety $\langle P_0\rangle = \gamma_0 H(\mathbb{R})^+ x_0$.

We claim that there exists a positive dimensional intersection component of $\pi^{-1}(V)$ containing $z_0 := \beta(0)$. To see this, let W denote the union of the totally geodesic subvarieties $\gamma H(\mathbb{R})^+ x$ of X, where (γ, x) varies over $\mathrm{Im}(\beta_1)$, and let $\overline{W}$ denote the Zariski closure of W in $X^\vee$. The irreducible analytic components of $\overline{W} \cap \pi^{-1}(V)$ are, by definition, intersection components of $\pi^{-1}(V)$. Since β_2 is continuous, it follows from the Global Decomposition Theorem (see [15], p172) that there exists $0 < t \leq 1$ and an intersection component A of $\pi^{-1}(V)$ that contains $\beta_2([0,t])$, which proves the claim.

Let B denote a Zariski optimal intersection component of $\pi^{-1}(V)$ containing A such that

$$\delta_{\mathrm{Zar}}(B) \leq \delta_{\mathrm{Zar}}(A),$$

and let Z denote the Zariski closure of $\pi(B)$ in S. By the weak hyperbolic Ax-Schanuel conjecture, $\langle B \rangle_{\mathrm{Zar}}$ is pre-weakly special and, as in the proof of Proposition 6.10,

$$\langle Z \rangle_{\mathrm{ws}} = \pi(\langle B \rangle_{\mathrm{Zar}}).$$

Therefore, we have $\dim Z \geq 1$ and, also,

$$\dim Z \geq \dim B \geq \dim \langle B \rangle_{\mathrm{Zar}} - \delta_{\mathrm{Zar}}(A) \geq \dim \langle Z \rangle_{\mathrm{ws}} - (\dim \overline{W} - 1),$$

where we use the fact that $\delta_{\mathrm{Zar}}(A)$ is at most $\dim \overline{W} - 1$. We claim that $\dim \overline{W} - 1 \leq \dim X_0$, which would conclude the proof as

$$\dim X_0 \leq \dim S - \dim V - 1$$

and this would imply that $Z \in \mathrm{an}(V)$, which is not allowed as $P_0 \in Z$.

Therefore, it remains to prove the claim. However, this is easy to prove working with complex duals and using the methods explained in the proof of Theorem 14.2.

□

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Christopher Daw
Department of Mathematics and Statistics,
University of Reading,
Whiteknights,
PO Box 217,
Reading,
Berkshire RG6 6AH,
United Kingdom.
E-mail address: chris.daw@reading.ac.uk

Jinbo Ren
Institut des Hautes Études Scientifiques,
Le Bois-Marie 35, route de Chartres,
91440 Bures-sur-Yvette, France.
E-mail address: renjinbo@ihes.fr